Financial Mathematics for Actuaries (Third Edition)

Chapter 3

Spot Rates, Forward Rates and the Term Structure

Learning Objectives

1. Spot rate of interest
2. Forward rate of interest
3. Yield curve
4. Term structure of interest rates
5. Interest rate swap
6. Swap rate

3.1 Spot and Forward Rates of Interest

- We now allow the rate of interest to vary with the duration of the investment.

- We consider the case where investments over different horizons earn different rates of interest, although the principle of compounding still applies.

- We consider two notions of interest rates, namely, the **spot rate of interest** and the **forward rate of interest**.

- Consider an investment at time 0 earning interest over t periods. We assume that the period of investment is fixed at the time of investment, but the rate of interest earned per period varies according to the investment horizon.

- Thus, we define i_t^S as the spot rate of interest, which is the annualized effective rate of interest for the period from time 0 to t.

- The subscript t in i_t^S highlights that the annual rate of interest varies with the investment horizon.

- Hence, a unit payment at time 0 accumulates to

$$a(t) = (1 + i_t^S)^t \tag{3.1}$$

at time t.

- The present value of a unit payment due at time t is

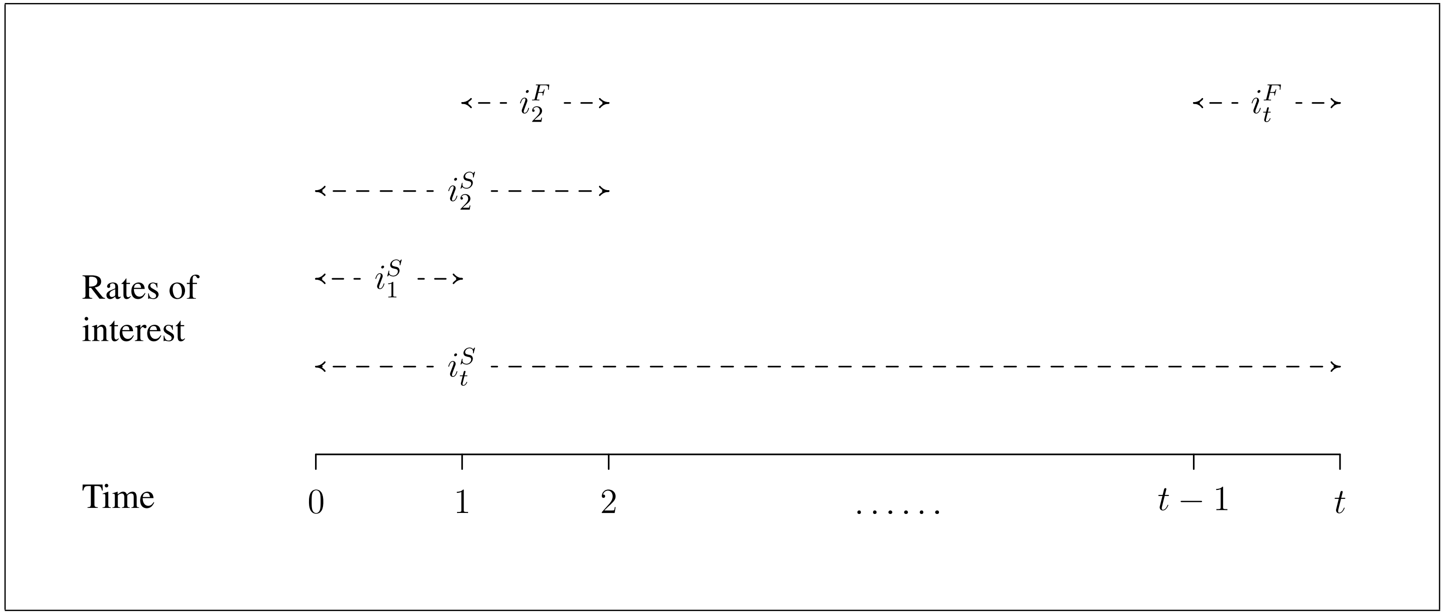
$$\frac{1}{a(t)} = \frac{1}{(1 + i_t^S)^t}. \tag{3.2}$$

- We define i_t^F as the rate of interest applicable to the period $t-1$ to t, called the forward rate of interest.

- This rate is determined at time 0, although the payment is due at time $t-1$ (thus, the use of the term *forward*).

- By convention, we have $i_1^S \equiv i_1^F$. However, i_t^S and i_t^F are generally different for $t = 2, 3, \cdots$. See Figure 3.1 for illustration.

Figure 3.1: Spot and forward rates of interest



- A plot of i_t^S against t is called the **yield curve**, and the mathematical relationship between i_t^S and t is called the **term structure of interest rates**.

- The spot and forward rates are not free to vary independently of each other.

- Consider the case of $t = 2$. If an investor invests a unit amount at time 0 over 2 periods, the investment will accumulate to $(1 + i_2^S)^2$ at time 2.

- Alternatively, she can invest a unit payment at time 0 over 1 period, and enters into a forward agreement to invest $1 + i_1^S$ unit at time 1 to earn the forward rate of i_2^F for 1 period.

- This rollover strategy will accumulate to $(1 + i_1^S)(1 + i_2^F)$ at time 2. The two strategies will accumulate to the same amount at time 2, so that

$$(1 + i_2^S)^2 = (1 + i_1^S)(1 + i_2^F), \tag{3.3}$$

if the capital market is *perfectly competitive*, so that no arbitrage opportunities exist.

- Equation (3.3) can be generalized to the following relationship concerning spot and forward rates of interest

$$(1 + i_t^S)^t = (1 + i_{t-1}^S)^{t-1}(1 + i_t^F), \tag{3.4}$$

for $t = 2, 3, \cdots$.

- We can also conclude that

$$(1 + i_t^S)^t = (1 + i_1^F)(1 + i_2^F) \cdots (1 + i_t^F). \tag{3.5}$$

- Given i_t^S, the forward rates of interest i_t^F satisfying equations (3.4) and (3.5) are called the *implicit* forward rates.

- The *quoted* forward rates in the market may differ from the implicit forward rates in practice, as when the market is noncompetitive.

- Unless otherwise stated we shall assume that equations (3.4) and (3.5) hold, so that it is the implicit forward rates we are referring to in our discussions.

- From equation (3.4),

$$i_t^F = \frac{(1 + i_t^S)^t}{(1 + i_{t-1}^S)^{t-1}} - 1. \tag{3.6}$$

Example 3.1: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate the forward rates of interest for $t = 1$, 2, 3 and 4.

Solution: First, $i_1^F = i_1^S = 4\%$. The rest of the calculation, using (3.6), is as follows

$$i_2^F = \frac{(1+i_2^S)^2}{1+i_1^S} - 1 = \frac{(1.045)^2}{1.04} - 1 = 5.0024\%,$$

$$i_3^F = \frac{(1+i_3^S)^3}{(1+i_2^S)^2} - 1 = \frac{(1.045)^3}{(1.045)^2} - 1 = 4.5\%$$

and

$$i_4^F = \frac{(1+i_4^S)^4}{(1+i_3^S)^3} - 1 = \frac{(1.05)^4}{(1.045)^3} - 1 = 6.5144\%.$$

$\square$

Example 3.2: Suppose the forward rates of interest for investments in year 1, 2, 3 and 4 are, respectively, 4%, 4.8%, 4.8% and 5.2%. Calculate the spot rates of interest for $t = 1$, 2, 3 and 4.

Solution: First, $i_1^S = i_1^F = 4\%$. From (3.5) the rest of the calculation is as follows

$$i_2^S = \left[(1+i_1^F)(1+i_2^F)\right]^{\frac{1}{2}} - 1 = \sqrt{1.04 \times 1.048} - 1 = 4.3992\%,$$

$$i_3^S = \left[(1+i_1^F)(1+i_2^F)(1+i_3^F)\right]^{\frac{1}{3}} - 1 = (1.04 \times 1.048 \times 1.048)^{\frac{1}{3}} - 1 = 4.5327\%$$

and

$$\begin{aligned} i_4^S &= \left[(1+i_1^F)(1+i_2^F)(1+i_3^F)(1+i_4^F)\right]^{\frac{1}{4}} - 1 \\ &= (1.04 \times 1.048 \times 1.048 \times 1.052)^{\frac{1}{4}} - 1 \\ &= 4.6991\%. \end{aligned}$$

$\square$

- We define the multi-period forward rate $i_{t,\tau}^F$ as the annualized rate of interest applicable over τ periods from time t to $t+\tau$, for $t \geq 1$ and $\tau > 0$, with the rate being determined at time 0.

- The following no-arbitrage relationships hold

$$(1+i^F_{t,\tau})^\tau = (1+i^F_{t+1})(1+i^F_{t+2})\cdots(1+i^F_{t+\tau}), \quad \text{for } t \geq 1, \tau > 0, \quad (3.7)$$

and

$$(1+i^S_{t+\tau})^{t+\tau} = (1+i^S_t)^t(1+i^F_{t,\tau})^\tau, \quad \text{for } t \geq 1, \tau > 0. \quad (3.8)$$

Example 3.3: Based on the spot rates of interest in Example 3.1, calculate the multi-period forward rates of interest $i^F_{1,2}$ and $i^F_{1,3}$.

Solution: Using (3.7) we obtain

$$i^F_{1,2} = [(1+i^F_2)(1+i^F_3)]^{\frac{1}{2}} - 1 = (1.050024 \times 1.045)^{\frac{1}{2}} - 1 = 4.7509\%.$$

Similarly, we have

$$i^F_{1,3} = (1.050024 \times 1.045 \times 1.065144)^{\frac{1}{3}} - 1 = 5.3355\%.$$

We may also use (3.8) to compute the multi-period forward rates. Thus,

$$i_{1,2}^F = \left[\frac{(1+i_3^S)^3}{1+i_1^S}\right]^{\frac{1}{2}} - 1 = \left[\frac{(1.045)^3}{1.04}\right]^{\frac{1}{2}} - 1 = 4.7509\%,$$

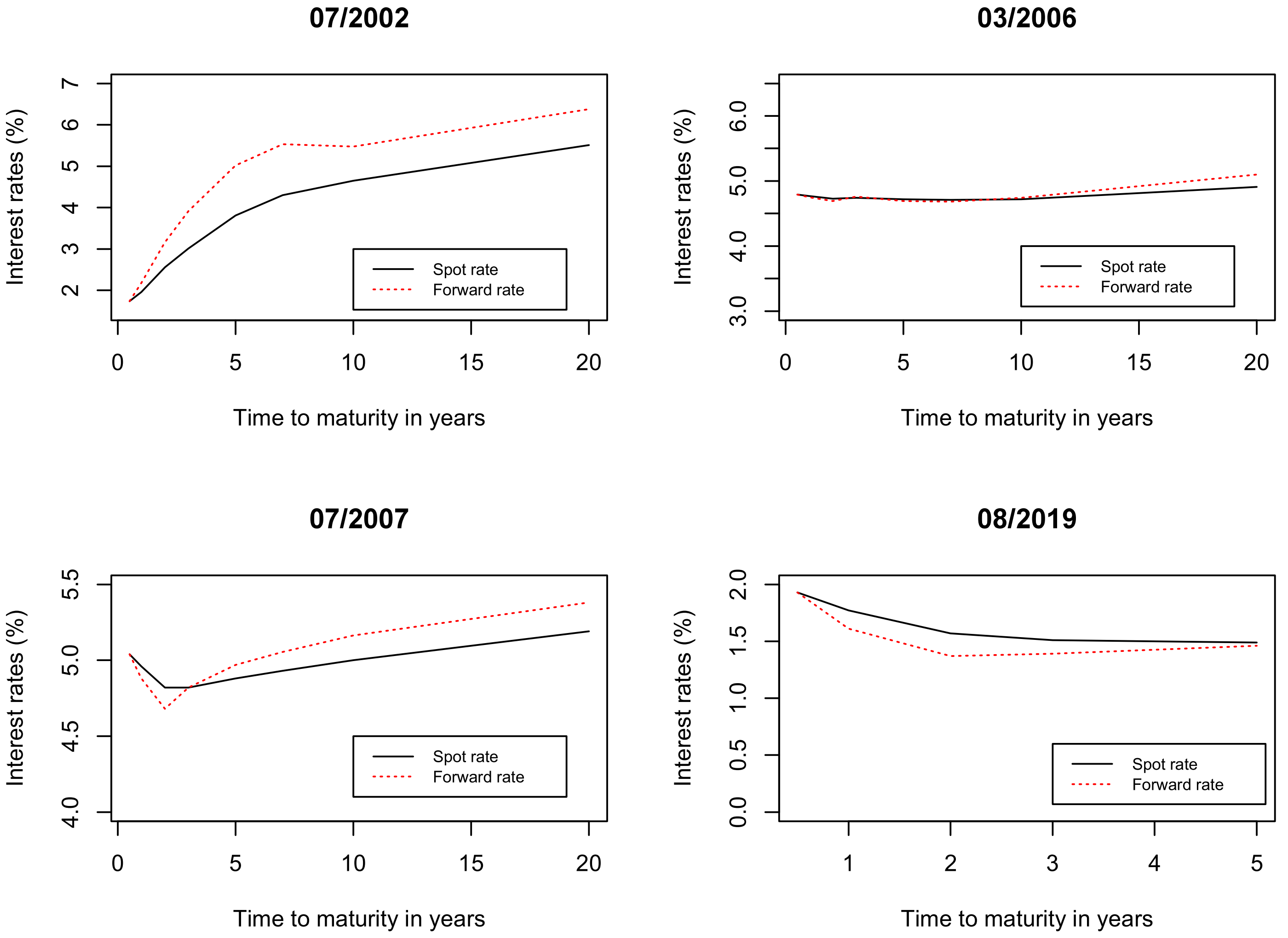
and similarly,

$$i_{1,3}^F = \left[\frac{(1+i_4^S)^4}{1+i_1^S}\right]^{\frac{1}{3}} - 1 = \left[\frac{(1.05)^4}{1.04}\right]^{\frac{1}{3}} - 1 = 5.3355\%.$$

3.2 The Term Structure of Interest Rates

- Empirically the term structure can take various shapes. A sample of four yield curves of the US market are presented in Figure 3.4.
- The spot-rate curve on 03/2006 is an example of a nearly **flat term structure**.
- On 08/2019 we have a **downward sloping term structure**. In this case the forward rate drops below the spot rate.
- We have an **upward sloping term structure** on 07/2002.
- Unlike the case of a downward sloping yield curve, the forward rate exceeds the spot rate when the yield curve is upward sloping.
- The yield curve on 07/2007 is **inverted humped**.

Figure 3.4: Yield curves of the US market



- An upward sloping yield curve is also said to have a **normal term structure** as this is the most commonly observed term structure empirically.

- We have an inverted **humped yield curve** on 2007/12/31.

- Some questions may arise from a cursory examination of this sample of yield curves. For example,

  – How are the yield curves obtained empirically?

  – What determines the shape of the term structure?

  – Why are upward sloping yield curves observed more often?

  – Does the term structure have any useful information about the real economy?

3.3 Present and Future Values Given the Term Structure

- We now consider the present and future values of an annuity given the term structure.

- We continue to use the actuarial notations introduced in Chapter 2.

- The present value of a unit-payment annuity-immediate over n periods is

$$
\begin{aligned}
a_{\overline{n}|} &= \sum_{t=1}^{n} \frac{1}{a(t)} \\
&= \sum_{t=1}^{n} \frac{1}{(1+i_t^S)^t} \\
&= \frac{1}{(1+i_1^S)} + \frac{1}{(1+i_2^S)^2} + \cdots + \frac{1}{(1+i_n^S)^n}.
\end{aligned} \tag{3.9}
$$

- The future value at time t of a unit payment at time 0 is

$$a(t) = (1 + i_t^S)^t = \prod_{j=1}^{t}(1 + i_j^F) = (1 + i_1^F)(1 + i_2^F)\cdots(1 + i_t^F). \quad (3.10)$$

- The present value of a unit payment due at time t is

$$\frac{1}{a(t)} = \frac{1}{\prod_{j=1}^{t}(1 + i_j^F)}, \quad (3.11)$$

and the present value of a n-payment annuity-immediate can also be written as

$$
\begin{aligned}
a_{\overline{n}|} &= \sum_{t=1}^{n} \frac{1}{a(t)} \\
&= \sum_{t=1}^{n} \frac{1}{\prod_{j=1}^{t}(1+i_j^F)} \\
&= \frac{1}{(1+i_1^F)} + \frac{1}{(1+i_1^F)(1+i_2^F)} + \cdots + \frac{1}{(1+i_1^F)\cdots(1+i_n^F)}.
\end{aligned}
\tag{3.12}
$$

- The computation of the future value at time n of a payment due at time t, where $0 < t < n$, requires additional assumptions.

- We consider the assumption that a **payment occurring in the future earns the forward rates of interest**.

- The future value at time n of a unit payment due at time t is

$$(1 + i^F_{t,n-t})^{n-t} = (1 + i^F_{t+1}) \cdots (1 + i^F_n), \tag{3.13}$$

and the future value at time n of a n-period annuity-immediate is

$$\begin{aligned} s_{\overline{n}\rceil} &= (1 + i^F_{1,n-1})^{n-1} + \cdots + (1 + i^F_{n-1,1}) + 1 \\ &= [(1 + i^F_2)(1 + i^F_3) \cdots (1 + i^F_n)] + [(1 + i^F_3)(1 + i^F_4) \cdots (1 + i^F_n)] + \cdots \\ &\quad \cdots + (1 + i^F_n) + 1. \end{aligned} \tag{3.14}$$

- From (3.14) we can see that

$$s_{\overline{n}\rceil} = \left[\prod_{t=1}^{n}(1 + i^F_t)\right] \times \left[\frac{1}{1 + i^F_1} + \frac{1}{(1 + i^F_1)(1 + i^F_2)} + \cdots + \frac{1}{(1 + i^F_1)\cdots(1 + i^F_n)}\right]. \tag{3.16}$$

- Hence,

$$s_{\overline{n}|} = \left(\prod_{t=1}^{n} (1 + i_t^F) \right) a_{\overline{n}|} = a(n) a_{\overline{n}|}. \tag{3.17}$$

- An alternative formula to calculate $s_{\overline{n}|}$ using the spot rates of interest is

$$\begin{aligned} s_{\overline{n}|} &= (1 + i_n^S)^n \, a_{\overline{n}|} \\ &= (1 + i_n^S)^n \left[\sum_{t=1}^{n} \frac{1}{(1 + i_t^S)^t} \right]. \end{aligned} \tag{3.18}$$

Example 3.4: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate $a_{\overline{4}|}$, $s_{\overline{4}|}$, $a_{\overline{3}|}$ and $s_{\overline{3}|}$.

Solution: From (3.9) we have

$$a_{\overline{4}|} = \frac{1}{1.04} + \frac{1}{(1.045)^2} + \frac{1}{(1.045)^3} + \frac{1}{(1.05)^4} = 3.5763.$$

From (3.17) we obtain

$$s_{\overline{4}|} = (1.05)^4 \times 3.5763 = 4.3470.$$

Similarly,

$$a_{\overline{3}|} = \frac{1}{1.04} + \frac{1}{(1.045)^2} + \frac{1}{(1.045)^3} = 2.7536,$$

and

$$s_{\overline{3}|} = (1.045)^3 \times 2.7536 = 3.1423.$$

□

Example 3.5: Suppose the 1-period forward rates of interest for investments due at time 0, 1, 2 and 3 are, respectively, 4%, 4.8%, 4.8% and 5.2%. Calculate $a_{\overline{4}|}$ and $s_{\overline{4}|}$.

Solution: From (3.12) we have

$$\begin{aligned} a_{\overline{4}|} &= \frac{1}{1.04} + \frac{1}{1.04 \times 1.048} + \frac{1}{1.04 \times 1.048 \times 1.048} + \frac{1}{1.04 \times 1.048 \times 1.048 \times 1.052} \\ &= 3.5867. \end{aligned}$$

As $a(4) = 1.04 \times 1.048 \times 1.048 \times 1.052 = 1.2016$, we have

$$s_{\overline{4}|} = 1.2016 \times 3.5867 = 4.3099.$$

Alternatively, from (3.14) we have

$$\begin{aligned} s_{\overline{4}|} &= 1 + 1.052 + 1.048 \times 1.052 + 1.048 \times 1.048 \times 1.052 \\ &= 4.3099. \end{aligned}$$

$\square$

- To further understand (3.17), we write (3.13) as (see (3.8))

$$(1 + i^F_{t,n-t})^{n-t} = \frac{(1 + i^S_n)^n}{(1 + i^S_t)^t} = \frac{a(n)}{a(t)}. \tag{3.19}$$

- Thus, the future value of the annuity is, from (3.14) and (3.19),

$$\begin{aligned}
s_{\overline{n}|} &= \left[\sum_{t=1}^{n-1} (1 + i^F_{t,n-t})^{n-t}\right] + 1 \\
&= \sum_{t=1}^{n} \frac{a(n)}{a(t)} \\
&= a(n) \sum_{t=1}^{n} \frac{1}{a(t)} \\
&= a(n) a_{\overline{n}|}. \qquad (3.20)
\end{aligned}$$

- In Chapter 2 we assume that the current accumulation function $a(t)$ applies to all future payments.

- Under this assumption, if condition (1.35) holds, equation (3.20) is valid. However, for a given general term structure, we note that

$$a(n-t) = (1 + i^S_{n-t})^{n-t} \neq \frac{(1 + i^S_n)^n}{(1 + i^S_t)^t} = \frac{a(n)}{a(t)},$$

  so that condition (1.35) does not hold.

- Thus, if future payments are assumed to earn spot rates of interest based on the current term structure, equation (3.20) does not hold in general.

Example 3.6: Suppose the spot rates of interest for investment horizons of 1 to 5 years are 4%, and for 6 to 10 years are 5%. Calculate the present value of an annuity-due of \$100 over 10 years. Compute the future value of the annuity at the end of year 10, assuming (a) future payments earn forward rates of interest, and (b) future payments earn the spot rates of interest as at time 0.

Solution: We consider the 10-period annuity-due as the sum of an annuity-due for the first 6 years and a deferred annuity-due of 4 payments starting at time 6. The present value of the annuity-due for the first 6 years is

$$
\begin{aligned}
100 \times \ddot{a}_{\overline{6}\rceil 0.04} &= 100 \times \left[\frac{1-(1.04)^{-6}}{1-(1.04)^{-1}}\right] \\
&= \$545.18.
\end{aligned}
$$

The present value at time 0 for the deferred annuity-due in the last 4 years is

$$
\begin{aligned}
100 \times (\ddot{a}_{\overline{10}\rceil 0.05} - \ddot{a}_{\overline{6}\rceil 0.05}) &= 100 \times \left[\frac{1-(1.05)^{-10}}{1-(1.05)^{-1}} - \frac{1-(1.05)^{-6}}{1-(1.05)^{-1}}\right] \\
&= 100 \times (8.1078 - 5.3295) \\
&= \$277.83.
\end{aligned}
$$

Hence, the present value of the 10-period annuity-due is

$$
545.18 + 277.83 = \$823.01.
$$

We now consider the future value of the annuity at time 10. Under assumption (a) that future payments earn the forward rates of interest, the future value of the annuity at the end of year 10 is, by equation (3.20),

$$
(1.05)^{10} \times 823.01 = \$1{,}340.60.
$$

Note that using (3.20) we do not need to compute the forward rates of interest to determine the future value of the annuity, as would be required if (3.16) is used.

Based on assumption (b), the payments at time $0, \cdots, 4$ earn interest at 5% per year (the investment horizons are 10 to 6 years), while the payments at time $5, \cdots, 9$ earn interest at 4% per year (the investment horizons are 5 to 1 years). Thus, the future value of the annuity is

$$100 \times (\ddot{s}_{\overline{10}\rceil 0.05} - \ddot{s}_{\overline{5}\rceil 0.05}) + 100 \times \ddot{s}_{\overline{5}\rceil 0.04} = \$1{,}303.78.$$

Thus, equation (3.20) does not hold under assumption (b). $\square$

3.4 Accumulation Function and the Term Structure

- Equation (3.1) can be extended to any $t > 0$, which need not be an integer.

- The annualized spot rate of interest for time to maturity t, i^S_t, is given by

$$i^S_t = [a(t)]^{\frac{1}{t}} - 1. \tag{3.22}$$

- We may also use an accumulation function to define the forward rates of interest.

- Let us consider the forward rate of interest for a payment due at time $t > 0$, and denote the accumulation function of this payment by $a_t(\cdot)$, where $a_t(0) = 1$.

- Now a unit payment at time 0 accumulates to $a(t+\tau)$ at time $t+\tau$, for $\tau > 0$.

- On the other hand, a strategy with an initial investment over t periods and a rollover at the forward rate for the next τ periods will accumulate to $a(t)a_t(\tau)$ at time $t+\tau$.

- By the no-arbitrage argument, we have

$$a(t)a_t(\tau) = a(t+\tau), \tag{3.23}$$

  so that

$$a_t(\tau) = \frac{a(t+\tau)}{a(t)}. \tag{3.24}$$

- The annualized forward rate of interest in the period t to $t+\tau$, $i^F_{t,\tau}$, satisfies

$$a_t(\tau) = (1 + i^F_{t,\tau})^\tau,$$

so that

$$i^F_{t,\tau} = [a_t(\tau)]^{\frac{1}{\tau}} - 1. \tag{3.25}$$

- If $\tau < 1$, we define the forward rate of interest per unit time (year) for the fraction of a period t to $t + \tau$ as

$$i^F_{t,\tau} = \frac{1}{\tau} \times \frac{a_t(\tau) - a_t(0)}{a_t(0)} = \frac{a_t(\tau) - 1}{\tau}. \tag{3.26}$$

- The **instantaneous forward rate of interest** per unit time at time t is equal to $i^F_{t,\tau}$ for $\tau \to 0$, which is given by

$$
\begin{aligned}
\lim_{\tau\to 0} i^F_{t,\tau} &= \lim_{\tau\to 0} \frac{a_t(\tau) - 1}{\tau} \\
&= \lim_{\tau\to 0} \frac{1}{\tau} \times \left[\frac{a(t+\tau)}{a(t)} - 1\right] \\
&= \frac{1}{a(t)} \lim_{\tau\to 0} \left[\frac{a(t+\tau) - a(t)}{\tau}\right] \\
&= \frac{a'(t)}{a(t)} \\
&= \delta(t). \qquad (3.27)
\end{aligned}
$$

- Thus, the instantaneous forward rate of interest per unit time is equal to the force of interest.

Example 3.7: Suppose $a(t) = 0.01t^2 + 0.1t + 1$. Compute the spot rates of interest for investments of 1, 2 and 2.5 years. Derive the accumulation function for payments due at time 2, assuming the payments earn the forward rates of interest. Calculate the forward rates of interest for time to maturity of 1, 2 and 2.5 years.

Solution: Using (3.22), we obtain $i_1^S = 11\%$, $i_2^S = 11.36\%$ and $i_{2.5}^S = 11.49\%$. Thus, we have an upward sloping spot-rate curve. To calculate the accumulation function of payments at time 2 we first compute $a(2)$ as

$$a(2) = 0.01(2)^2 + 0.1(2) + 1 = 1.24.$$

Thus, the accumulation function for payments at time 2 is

$$a_2(t) = \frac{a(2+t)}{a(2)} = \frac{0.01(2+t)^2 + 0.1(2+t) + 1}{1.24} = 0.0081t^2 + 0.1129t + 1.$$

Using the above equation, we obtain $a_2(1) = 1.1210$, $a_2(2) = 1.2581$ and $a_2(2.5) = 1.3327$, from which we conclude $i_{2,1}^F = 12.10\%$,

$$i_{2,2}^F = (1.2581)^{\frac{1}{2}} - 1 = 12.16\%$$

and

$$i^F_{2,2.5} = (1.3327)^{\frac{1}{2.5}} - 1 = 12.17\%.$$

Thus, the forward rates exceed the spot rates, which agrees with what might be expected of an upward sloping yield curve. □

- We can further establish the relationship between the force of interest and the forward accumulation function, and thus the forward rates of interest.

- From (3.24), we have

$$a_t(\tau) = \frac{a(t+\tau)}{a(t)} = \frac{\exp\left(\int_0^{t+\tau} \delta(s)\, ds\right)}{\exp\left(\int_0^{t} \delta(s)\, ds\right)} = \exp\left(\int_t^{t+\tau} \delta(s)\, ds\right), \tag{3.28}$$

  so that we can compute the forward accumulation function from the force of interest.

Example 3.8: Suppose $\delta(t) = 0.05t$. Derive the accumulation function for payments due at time 2, assuming the payments earn the forward rates of interest. Calculate the forward rates of interest for time to maturity of 1 and 2 years.

Solution: Using (3.26) we obtain

$$a_2(t) = \exp\left(\int_2^{2+t} 0.05s\, ds\right) = \exp\left[0.025(2+t)^2 - 0.025(2)^2\right] = \exp(0.025t^2 + 0.1t).$$

Thus, we can check that $a_2(0) = 1$. Now,

$$a_2(1) = \exp(0.125) = 1.1331,$$

so that $i^F_{2,1} = 13.31\%$. Also, $a_2(2) = \exp(0.3) = 1.3499$, so that

$$i^F_{2,2} = (1.3499)^{\frac{1}{2}} - 1 = 16.18\%.$$

$\square$

- We now consider payments of $C_1, C_2, \cdots, C_n$ at time $t_1 < t_2 < \cdots < t_n$, respectively.

- We wish to compute the value of these cash flows at any time t (≥ 0).

- For the payment C_j at time $t_j \leq t$, its accumulated value at time t is $C_j\, a_{t_j}(t - t_j)$. On the other hand, if $t_j > t$, the discounted value of C_j at time t is $C_j/a_t(t_j - t)$.

- Thus, the value of the cash flows at time t is (see equation (3.24))

$$
\begin{aligned}
\sum_{t_j \le t} C_j \, a_{t_j}(t - t_j) + \sum_{t_j > t} C_j \left[\frac{1}{a_t(t_j - t)} \right] &= \sum_{t_j \le t} C_j \left[\frac{a(t)}{a(t_j)} \right] + \sum_{t_j > t} C_j \left[\frac{a(t)}{a(t_j)} \right] \\
&= \sum_{j=1}^{n} C_j \left[\frac{a(t)}{a(t_j)} \right] \\
&= a(t) \sum_{j=1}^{n} C_j \, v(t_j) \\
&= a(t) \times \text{present value of cash flows.}
\end{aligned}
\tag{3.29}
$$

- An analogous result can be obtained if we consider a continuous cash flow.

- If $C(t)$ is the instantaneous rate of cash flow at time t for $0 \le t \le n$, the value of the cash flow at time $\tau \in [0, n]$ is

$$\begin{aligned}
\int_0^\tau C(t)a_t(\tau - t)\,dt + \int_\tau^n \frac{C(t)}{a_\tau(t-\tau)}\,dt &= \int_0^\tau \frac{C(t)a(\tau)}{a(t)}\,dt + \int_\tau^n \frac{C(t)a(\tau)}{a(t)}\,dt \\
&= a(\tau)\int_0^n \frac{C(t)}{a(t)}\,dt \\
&= a(\tau)\int_0^n C(t)v(t)\,dt. \qquad (3.30)
\end{aligned}$$

Example 3.9: Suppose $a(t) = 0.02t^2 + 0.05t + 1$. Calculate the value at time 3 of a 1-period deferred annuity-immediate of 4 payments of \$2 each. You may assume that future payments earn the forward rates of interest.

Solution: We first compute the present value of the annuity. The payments of \$2 are due at time 2, 3, 4 and 5. Thus, the present value of the cash flows is

$$2 \times \left[\frac{1}{a(2)} + \frac{1}{a(3)} + \frac{1}{a(4)} + \frac{1}{a(5)} \right].$$

Now, $a(2) = 0.02(2)^2 + 0.05(2) + 1 = 1.18$, and similarly we have $a(3) = 1.33$, $a(4) = 1.52$ and $a(5) = 1.75$. Thus, the present value of the cash flow is

$$2 \times \left[\frac{1}{1.18} + \frac{1}{1.33} + \frac{1}{1.52} + \frac{1}{1.75} \right] = 2 \times 2.82866 = \$5.6573,$$

and the value of the cash flow at time 3 is $a(3) \times 5.6573 = 1.33 \times 5.6573 = \$7.5242$. □

Example 3.10: Suppose $\delta(t) = 0.02t$. An investor invests in a fund at the rate of $10t$ per period at time t, for $t > 0$. How much would she accumulate in the fund at time 2? You may assume that future payments earn the forward rates of interest.

Solution: The amount she invests in the period $(t, t+\Delta t)$ is $10t\Delta t$, which would accumulate to $(10t\Delta t)a_t(2-t)$ at time 2. Thus, the total amount accumulated at time 2 is

$$\int_0^2 10ta_t(2-t)dt.$$

From (3.28), we have

$$a_t(2-t) = \exp\left(\int_t^2 \delta(s)\,ds\right) = \exp\left(\int_t^2 0.02s\,ds\right).$$

Now, we have

$$\int_t^2 0.02s\,ds = 0.01s^2]_t^2 = 0.01(2)^2 - 0.01t^2,$$

so that

$$a_t(2-t) = \exp(0.04 - 0.01t^2)$$

and

$$\begin{aligned}
\int_0^2 10ta_t(2-t)dt &= 10\int_0^2 te^{0.04-0.01t^2}\,dt \\
&= 10e^{0.04}\int_t^2 te^{-0.01t^2}\,dt \\
&= \frac{10e^{0.04}}{0.02}\left(-e^{-0.01t^2}\big]_0^2\right) \\
&= \frac{10e^{0.04}(1-e^{-0.04})}{0.02} \\
&= 20.4054.
\end{aligned}$$

□

Example 3.11: Suppose the principal is C and interest is earned at the force of interest $\delta(t)$, for $t > 0$. What is the present value of the interest earned over n periods.

Solution: As $\delta(t)$ is the instantaneous rate of interest per period at time t, the amount of interest earned in the period $(t, t+\Delta t)$ is $C\delta(t)\Delta t$, and the present value of this interest is $[C\delta(t)\Delta t]v(t)$. Thus, the present value of all the interest earned in the period $(0, n)$ is

$$\int_0^n C\delta(t)v(t)\,dt.$$

Now, we have

$$\begin{aligned}
\int_0^n \delta(t)v(t)\,dt &= \int_0^n \delta(t)\exp\left(-\int_0^t \delta(s)\,ds\right)dt \\
&= \left(-\exp\left(-\int_0^t \delta(s)\,ds\right)\right)\Big]_0^n \\
&= \exp\left(-\int_0^0 \delta(s)\,ds\right) - \exp\left(-\int_0^n \delta(s)\,ds\right) \\
&= 1 - v(n).
\end{aligned}$$

Hence, the present value of the interest earned is

$$C[1 - v(n)] = C - Cv(n),$$

which is the principal minus the present value of the principal redeemed at time n. $\square$

3.5 Interest Rate Swaps

- An interest rate swap is an agreement between two parties to exchange cash flows based on interest rate movements.

- A simple interest rate swap arrangement involves two companies. Company A agrees to pay cash flows to Company B equal to the amount of interest at a pre-fixed rate on a notional principal amount for a number of years.

- At the same time, Company B consents to pay interests to Company A at a floating rate on the same notional principal for the same period of time.

- The following features of an interest rate swap are often agreed upon at the issue date.

Swap Term (or Swap Tenor): The contract period of the interest rate swap. It may be as short as a few months, or as long as 30 years.

Settlement Dates: The specified dates that two counterparties have to exchange interest payments.

Settlement Period: The time between settlement dates is called the settlement period. The settlement period specifies the frequency of interest payments. It can be annually, quarterly, monthly, or at any other interval determined by the parties.

Notional Amount: The notional principal amount is the predetermined dollar amount on which the exchanged interest payments are based. It is only used for the calculation of interest payments and is never exchanged.

Swap Rate: The fixed interest rate specified in the interest rate swap contract.

Floating Interest Rate: This is the reference rate for calculating the floating interest payment at each settlement date. It must be specified in the interest rate swap contract.

- In general the notional amount is a constant over the swap tenor. However, an **accreting swap** has the scheduled notional amount increasing over time, while an **amortizing swap** has the notional amount declining over time.
- Commonly used floating reference interest rates in a swap agreement include: Treasury Bill Rate, Prime Rate, Federal Funds Rate in the US market, and the LIBOR (London Interbank Offer Rate) in the international markets.

- The floating rate can be specified with a spread to the reference rate.

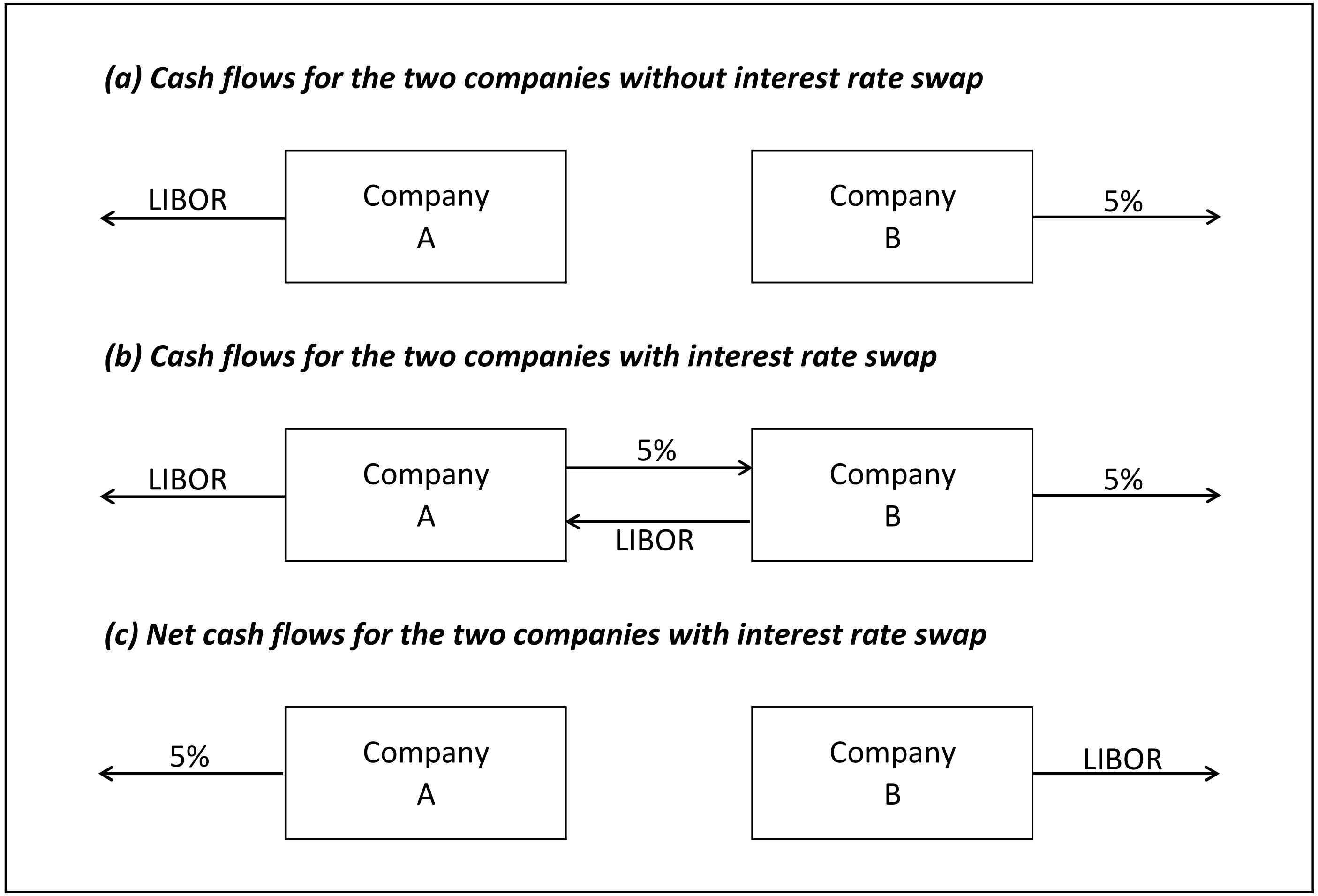
- Interest rate swaps can be used to convert a floating rate loan to a fixed rate debt or vice versa.

- Suppose Company A has arranged to borrow US$10 million for five years. The amount of interest is payable at the end of each settlement year according to the 12-month US dollar LIBOR at the beginning of the settlement period.

- Company A, however, would like to transform this floating rate loan to a fixed one. On the other hand, Company B has arranged to borrow US$10 million at a fixed rate of 5% per annum for five years, but would prefer to convert this fixed rate loan to a floating rate loan.

- Company A and Company B may enter into a simple interest rate swap agreement, with Company A being the **fixed-rate payer** and Company B being the **floating-rate payer**. Figure 3.5 illustrates the arrangement of the swap.

Figure 3.5: Illustration of cash flows of an interest rate swap



- The fixed interest rate in the interest rate swap contract is called the **swap rate**, which will be denoted by R_S.

- We now discuss a theoretical framework for deriving R_S such that the swap contract is equally attractive to both parties.

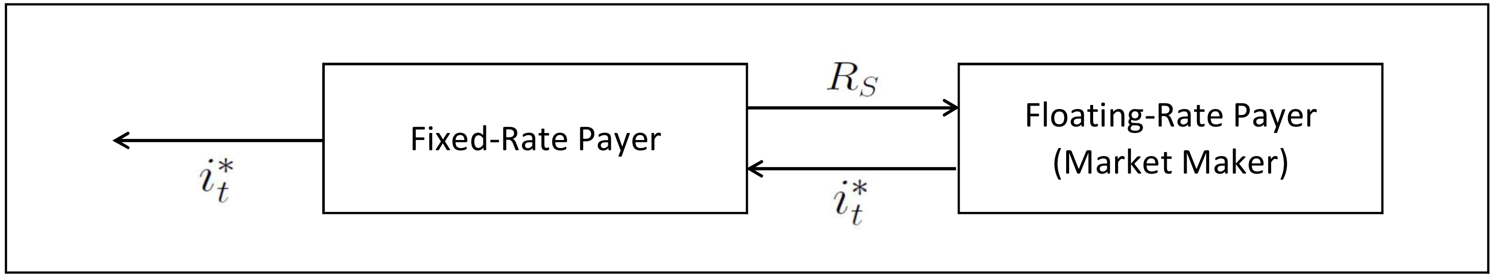
- Consider a n-year interest rate swap contract with notional amount m_t for year $t = 1, \cdots, n$. The settlement period is one year. At time 0, when the contract is initiated, the spot rates of interest i_t^S are known.

- The floating interest rate is defined as the **realized** one-year spot rate i_t^* observed at the beginning of the t^{th} future settlement period. It is obvious that $i_1^* = i_1^S$.

- Suppose a fixed-rate payer enters into a contract with the counterparty (i.e., the floating rate payer), which is the market maker.

Figure 3.6 illustrates the cash flows of these parties.

Figure 3.6: Cash flows of an interest rate swap



- At the end of year t the market maker will receive $m_t R_S$ and pays $m_t i_t^*$ so that his net receipt is $m_t(R_S - t\imath_t^*)$.

- However, he can enter into a forward contract to borrow m_t at time $t-1$ at forward rate i_t^F.

- He will then lend the borrowed amount at the prevailing spot rate i_t^* for one period.

- At time t the market maker's net receipt on the forward contract and the spot lending is $m_t(i_t^* - i_t^F)$.

- Table 3.1 summarizes the cash flows faced by the market maker.

Table 3.1: Cash flows of the market maker in Figure 3.6

Year	Net receipt on forward contract	Net receipt on swap contract	Overall
1	$m_1(i_1^* - i_1^F)$	$m_1(R_S - i_1^*)$	$m_1(R_S - i_1^F)$
2	$m_2(i_2^* - i_2^F)$	$m_2(R_S - i_2^*)$	$m_2(R_S - i_2^F)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	$m_t(i_t^* - i_t^F)$	$m_t(R_S - i_t^*)$	$m_t(R_S - i_t^F)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$m_n(i_n^* - i_n^F)$	$m_n(R_S - i_n^*)$	$m_n(R_S - i_n^F)$

- Under the perfect and frictionless market assumption, the present value of the overall cash flows of the market maker should be zero.

- Therefore, we have

$$m_1\left[\frac{R_S - i_1^F}{(1+i_1^S)}\right] + m_2\left[\frac{R_S - i_2^F}{(1+i_2^S)^2}\right] + \cdots + m_n\left[\frac{R_S - i_n^F}{(1+i_n^S)^n}\right] = 0. \quad (3.31)$$

- Hence,

$$R_S = \frac{\displaystyle\sum_{t=1}^{n} m_t\ i_t^F (1+i_t^S)^{-t}}{\displaystyle\sum_{t=1}^{n} m_t\ (1+i_t^S)^{-t}}. \quad (3.32)$$

- For interest rate swaps with a constant notional principal amount for every year, equation (3.32) can be simplified to

$$R_S = \frac{\displaystyle\sum_{t=1}^{n} i_t^F (1 + i_t^S)^{-t}}{\displaystyle\sum_{t=1}^{n} (1 + i_t^S)^{-t}}. \tag{3.33}$$

Example 3.12: A company enters into a 5-year interest rate swap contract with a level notional amount of \$1 million. The settlement period is one year. The floating interest rate is defined as the realized one-year spot rate observed at the beginning of each settlement period. The spot rates of interest at the initiation of the swap for investment horizons of 1, 2, 3, 4 and 5 years are, respectively, 3.5%, 3.8%, 4.3%, 4.9% and 5.2%. Determine the swap rate.

Solution: The forward rates of interest i_t^F for $t = 1, 2, 3, 4$ and 5 can be calculated using equation (3.6). Table 3.2 shows the computation of the swap rate via equation (3.33). Hence,

$$R_S = \frac{0.223894}{4.377604} = 5.1145\%$$

Table 3.2: Computation results for Example 3.12

t	i_t^S	i_t^F	$i_t^F(1+i_t^S)^{-t}$	$(1+i_t^S)^{-t}$
1	0.035000	0.035000	0.033816	0.966184
2	0.038000	0.041009	0.038061	0.928122
3	0.043000	0.053072	0.046775	0.881347
4	0.049000	0.067208	0.055503	0.825844
5	0.052000	0.064086	0.049738	0.776106
Total			0.223894	4.377604

$\square$