

Financial Mathematics for Actuaries (Third Edition)

Chapter 7

Bond Yields and Term Structure

Learning Objectives

1. Yield to maturity, yield to call and par yield
2. Realized compound yield and horizon analysis
3. Estimation of the yield curve: bootstrap method and least squares method
4. Estimation of the instantaneous forward rate and the term structure
5. Models of the determination of the term structure

7.1 Some Simple Measures of Bond Yield

- **Current yield** is the annual dollar amount of coupon payment(s) divided by the quoted (clean) price of the bond.
- It is the annual coupon rate of interest divided by the price of the bond per unit face value.
- The current yield does not adequately reflect the potential gain of the bond investment.
- The **nominal yield** is the annual amount of coupon payment(s) divided by the face value, or simply the coupon rate of interest per annum. Like the current yield, the nominal yield is not a good measure of the potential return of the bond.

7.2 Yield to Maturity

- Given the yield rate applicable under the prevailing market conditions, we can compute the bond price using one of the pricing formulas in Chapter 6.
- In practice, however, the yield rate or the term structure are not *observable*, while the transaction price of the bond can be observed from the market.
- Given the transaction price, we can solve for the rate of interest that equates the discounted future cash flows (coupon payments and redemption value) to the transaction price.
- This rate of interest is called the **yield to maturity** (or the **yield to redemption**), which is the return on the bond investment if the

investor holds the bond until it matures, assuming all the entitled payments are realized.

- The yield to maturity is indeed the internal rate of return of the bond investment.
- For a n -year annual coupon bond with transaction price P , the yield to maturity, denote by i_Y , is the solution of the following equation:

$$P = Fr \sum_{j=1}^n \frac{1}{(1 + i_Y)^j} + \frac{C}{(1 + i_Y)^n}. \quad (7.1)$$

- In the case of a n -year semiannual coupon bond, we solve i_Y from the equation (the coupon rate r is now per half-year)

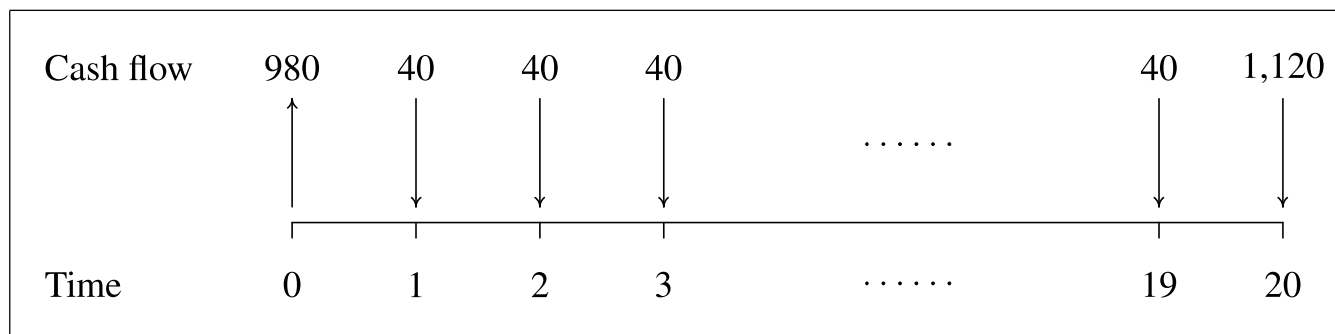
$$P = Fr \sum_{j=1}^{2n} \frac{1}{\left(1 + \frac{i_Y}{2}\right)^j} + \frac{C}{\left(1 + \frac{i_Y}{2}\right)^{2n}}. \quad (7.2)$$

- To calculate the solutions of equations (7.1) and (7.2) numerical methods are required. The Excel Solver may be used for the computation.

Example 7.1: A \$1,000 par value 10-year bond with redemption value of \$1,080 and coupon rate of 8% payable semiannually is purchased by an investor at the price of \$980. Find the yield to maturity of the bond.

Solution: The cash flows of the bond in this example are plotted in Figure 7.1.

Figure 7.1: Cash-flow diagram for Example 7.1



We solve for i from the following equation:

$$980 = 40a_{\overline{20}|i} + 1,080(1 + i)^{-20}$$

to obtain $i = 4.41\%$, so that the yield to maturity i_Y is 8.82% per annum convertible semiannually. □

Example 7.2: The following shows the information of a bond traded in the secondary market:

<i>Type of Bond</i>	<i>Non-callable government bond</i>
<i>Issue Date</i>	<i>March 10, 2002</i>
<i>Maturity Date</i>	<i>March 10, 2012</i>
<i>Face Value</i>	<i>\$100</i>
<i>Redemption Value</i>	<i>\$100</i>
<i>Coupon Rate</i>	<i>4% payable semiannually</i>

Assume that the coupon dates of the bond are March 10 and September 10 of each year. Investor A bought the bond on the issue date at a price of \$105.25. On January 5, 2010, Investor B purchased the bond from A at a purchase (dirty) price of \$104.75. Find (a) the yield to maturity of Investor A's purchase on March 10, 2002, (b) the realized yield to Investor A on the sale of the bond, and (c) the yield to maturity of Investor B's purchase on January 5, 2010.

Solution: (a) We need to solve for i in the basic price formula

$$P = (Fr)a_{\overline{n}|i} + Cv^n,$$

which, on the issue date, is

$$105.25 = 2a_{\overline{20}|i} + 100(1 + i)^{-20}.$$

The numerical solution of i is 1.69% (per half-year). Thus, the yield to maturity is 3.38% per annum convertible semiannually.

(b) Investor A received the 15th coupon payment on September 10, 2009 and there are 117 days between September 10, 2009 and the sale date of January 5, 2010. The number of days between the two coupon payments, namely, September 10, 2009 and March 10, 2010 is 181. Thus, we need to compute i in the following equation:

$$105.25 = 2a_{\overline{15}|i} + 104.75(1 + i)^{-15\frac{117}{181}},$$

the solution of which is 1.805% (per half-year). Thus, the realized yield is 3.61% per annum convertible semiannually.

(c) Investor B will receive the next 5 coupon payments starting on March 10, 2010. There are 64 days between the purchase date and the next coupon date. Investor B's return i per half-year is the solution of the following equation:

$$\begin{aligned} 104.75 &= [2\ddot{a}_{\overline{5}|i} + 100(1+i)^{-4}] (1+i)^{-\frac{64}{181}} \\ &= [2a_{\overline{5}|i} + 100(1+i)^{-5}] (1+i)^{\frac{117}{181}}, \end{aligned}$$

which is 1.18%. Thus, the yield to maturity is 2.36% per annum convertible semiannually. \square

- If a bond is callable prior to its maturity, a commonly quoted measure is the **yield to call**, which is computed in the same way as the yield to maturity, with the following modifications:
 - the redemption value in equations (7.1) and (7.2) is replaced by the call price,

- the maturity date is replaced by the call date.
- An investor will be able to compute a schedule of the yield to call as a function of the call date (which also determines the call price), and assess her investment over the range of possible yields.
- Although the Excel Solver can be used to calculate the solution of equations (7.1) and (7.2), the computation can be more easily done using the Excel function YIELD, the specification of which is given as follows:

Excel function: YIELD(smt,mty,crt,prc,rdv,frq,basis)

smt = settlement date

mty = maturity date

crt = coupon rate of interest per annum

prc = quoted (clean) price of the bond per 100 face value

rdv = redemption value per 100 face value

frq = number of coupon payments per year

basis = day count, 30/360 if omitted and actual/actual if set to 1

Output = yield to maturity of the bond

- Exhibit 7.1 illustrates the use of the function YIELD to solve Example 7.2.

Exhibit 7.1: Use of Excel function YIELD

	A	B	C	D	E
1	3/10/2002				
2	1/5/2010				
3	3/10/2012				
4	3.38%	= YIELD(A1 ,A3 ,0.04 ,105.25 ,100 ,2 ,1)			
5	2.36%	= YIELD(A2 ,A3 ,0.04 ,103.4572 ,100 ,2 ,1)			
6					

- We first enter the dates March 10, 2002, January 5, 2010 and March 10, 2012 into Cells A1 through A3, respectively.
- In Cell A4, the YIELD function is entered, with settlement date A1, maturity date A3 and price of bond 105.25.
- The output is the yield to maturity of Investor A's purchase on March 10, 2002, i.e., 3.38%, which is the answer to Part (a).
- Part (b) cannot be solved by the YIELD function, as the bond was not sold on a coupon-payment date.
- For Part (c), the input price of the bond required in the YIELD function is the quoted price. We first compute the quoted price of the bond on January 5, 2010, which is $104.75 - 2 \times \frac{117}{181} = 103.4572$.
- The answer to Part (c) is shown in Cell A5 to be 2.36%.

7.3 Par Yield

- Given the prevailing spot-rate curve and that bonds are priced according to the existing term structure, the yield to maturity i_Y is solved from the following equation (for an annual coupon bond):

$$Fr \sum_{j=1}^n \frac{1}{(1 + i_Y)^j} + \frac{C}{(1 + i_Y)^n} = Fr \sum_{j=1}^n \frac{1}{(1 + i_j^S)^j} + \frac{C}{(1 + i_n^S)^n}, \quad (7.3)$$

which is obtained from equations (6.10) and (7.1).

- Hence, i_Y is a nonlinear function *averaging* the spot rates i_j^S , $j = 1, \dots, n$.
- As an averaging measure of the spot rates, however, i_Y has some disadvantages.

- First, no analytic solution of i_Y exists.
 - Second, i_Y varies with the coupon rate of interest, even for bonds with the same maturity.
- To overcome these difficulties, the **par yield** may be used, which is defined as the coupon rate of interest such that the bond is traded at par based on the prevailing term structure.
 - Denoting the par yield by i_P and setting $F = C = 100$, we have

$$100 = 100 i_P \sum_{j=1}^n \frac{1}{(1 + i_j^S)^j} + \frac{100}{(1 + i_n^S)^n}, \quad (7.4)$$

from which we obtain

$$i_P = \frac{1 - (1 + i_n^S)^{-n}}{\sum_{j=1}^n (1 + i_j^S)^{-j}}. \quad (7.5)$$

- More generally, if the bond makes level coupon payments at time t_1, \dots, t_n , and the term structure is defined by the accumulation function $a(\cdot)$, the par yield is given by

$$\begin{aligned}
 i_P &= \frac{1 - [a(t_n)]^{-1}}{\sum_{j=1}^n [a(t_j)]^{-1}} \\
 &= \frac{1 - v(t_n)}{\sum_{j=1}^n v(t_j)}. \tag{7.6}
 \end{aligned}$$

- Table 7.2 illustrates the par yields computed from two different spot-rate curves: an upward sloping curve and a downward sloping curve.

Table 7.2: Par yields of two term structures

n	Case 1		Case 2	
	i_n^S	i_P	i_n^S	i_P
1	3.50	3.50	6.00	6.00
2	3.80	3.79	5.70	5.71
3	4.10	4.08	5.40	5.42
4	4.40	4.37	5.10	5.14
5	4.70	4.64	4.80	4.86
6	5.00	4.91	4.50	4.58
7	5.30	5.18	4.20	4.30
8	5.60	5.43	3.90	4.02
9	5.90	5.67	3.60	3.74
10	6.20	5.91	3.30	3.46
11	6.50	6.13	3.00	3.18
12	6.80	6.34	2.70	2.89

- The par yield is more a summary measure of the existing term structure than a measure of the potential return of a bond investment.
- On the other hand, the yield to maturity is an *ex ante* measure of the return of a bond.
- If the bond is sold before it matures, or if the **interest-on-interest** is different from the current yield rate, the *ex post* return of the bond will be different.
- We now consider the evaluation of the return of a bond investment taking account of the possibility of varying interest rates prior to redemption as well as sale of the bond before maturity.

7.4 Holding-Period Yield

- For various reasons, investors often sell their bonds before maturity.
- The **holding-period yield**, also called the **realized compound yield** or the **total return**, is often computed on an *ex post* basis to evaluate the average return of the investment over the holding period of the bond.
- This methodology can also be applied to assess the possible return of the bond investment over a targeted horizon under different interest-rate scenarios.
- This application, called **horizon analysis**, is a useful tool for active bond management.

- We denote the holding-period yield of a bond by i_H .
- We first discuss the simple case of a **one-period holding yield**.
- Suppose a bond is purchased at time $t - 1$ for P_{t-1} . At the end of the period the bondholder receives a coupon of Fr and then sells the bond for P_t .
- The holding yield over the period $t - 1$ to t , denoted by i_H , is then given by

$$i_H = \frac{(P_t + Fr) - P_{t-1}}{P_{t-1}}. \quad (7.7)$$

Example 7.4: Consider the bond in Table 6.1. Find the holding-period yield in the third half-year period. Assume that the yield rate is unchanged since the issue date and remains at 2% per half-year.

Solution: From the amortization schedule in Table 6.1, we have $P_2 = 1,019.04$, $P_3 = 1,014.42$ and coupon payment of 25. Therefore, the holding-period yield for the third half-year period is

$$i_H = \frac{(1,014.42 + 25.00) - 1,019.04}{1,019.04} = 2\%.$$

□

- Example 7.4 illustrates that when the yield rate is unchanged after a bond investment until it is sold prior to maturity, the holding-period yield in that period is equal to the yield to maturity.
- However, when the prevailing yield rate fluctuates, so will the bond's holding-period yield.
- The computation of the one-period holding yield can be extended to multiple periods.

- To calculate the n -period holding yield over n coupon-payment periods, we need to know the interest earned by the coupons when they are paid.
- Let P_0 be the beginning price of the bond, P_n be the ending price of the bond and V be the accumulated value of the n coupons at time n .
- The n -period holding yield i_H is then the solution of the equation

$$P_0(1 + i_H)^n = P_n + V, \quad (7.8)$$

so that the annualized n -period holding yield is

$$i_H = \left[\frac{P_n + V}{P_0} \right]^{\frac{1}{n}} - 1. \quad (7.9)$$

Example 7.5: Consider a \$1,000 face value 5-year non-callable bond with annual coupons of 5%. An investor bought the bond at its issue date at a price of \$980. After receiving the third coupon payment, the investor immediately sold the bond for \$1,050. The investor deposited all coupon income in a savings account earning an effective rate of 2% per annum. Find the annualized holding-period yield i_H of the investor.

Solution: We have $P_0 = 980$ and $P_3 = 1,050$. The accumulated interest-on-interest of the coupons, V , is given by

$$V = (1,000 \times 0.05) s_{\overline{3}|0.02} = 50 \times 3.0604 = 153.02.$$

Thus, the annualized 3-period holding yield i_H is the solution of

$$980(1 + i_H)^3 = 1,050 + 153.02 = 1,203.02,$$

which implies

$$i_H = \left[\frac{1,203.02}{980} \right]^{\frac{1}{3}} - 1 = 7.0736\%.$$

□

Example 7.6: Consider two bonds, A and B. Bond A is a 10-year 2% annual-coupon bond, and Bond B is a 3-year 4% annual-coupon bond. The current spot-rate curve is flat at 3%, and is expected to remain flat for the next 3 years. A fund manager assumes two scenarios of interest-rate movements. In Scenario 1, the spot rate increases by 0.25 percentage point each year for 3 years. In Scenario 2, the spot rate drops to 2% next year and remains unchanged for 2 years. If the manager has an investment horizon of 3 years, what is her recommended strategy under each scenario? You may assume that all coupons and their interests are reinvested to earn the prevailing one-year spot rate.

Solution: We first compute the current bond prices. For Bond A, the current price is

$$2a_{\overline{10}|0.03} + 100(1.03)^{-10} = 91.4698,$$

and for Bond B, its current price is

$$4a_{\overline{3}|0.03} + 100(1.03)^{-3} = 102.8286.$$

Under Scenario 1, the price of Bond A after 3 years is

$$2a_{\overline{7}|0.0375} + 100(1.0375)^{-7} = 89.3987,$$

and the accumulated value of the coupons is $2 \times (1.0325 \times 1.035 + 1.035 + 1) = 6.2073$. Thus, the holding-period yield of Bond A over the 3 years is

$$\left[\frac{89.3987 + 6.2073}{91.4698} \right]^{\frac{1}{3}} - 1 = 1.4851\%.$$

Under Scenario 2, Bond A is traded at par in year 3 with the accumulated value of the coupons being $2 \times (1.02 \times 1.02 + 1.02 + 1) = 6.1208$. Thus, the holding-period yield is

$$\left[\frac{100 + 6.1208}{91.4698} \right]^{\frac{1}{3}} - 1 = 5.0770\%.$$

On the other hand, Bond B matures in 3 years so that its holding-period yield under Scenario 1 is

$$\left[\frac{100 + 4 \times (1.0325 \times 1.035 + 1.035 + 1)}{102.8286} \right]^{\frac{1}{3}} - 1 = 3.0156\%,$$

while under Scenario 2 its holding-period yield is

$$\left[\frac{100 + 4 \times (1.02 \times 1.02 + 1.02 + 1)}{102.8286} \right]^{\frac{1}{3}} - 1 = 2.9627\%.$$

Thus, Bond B is the preferred investment under Scenario 1, while Bond A is preferred under Scenario 2. □

7.5 Discretely Compounded Yield Curve

- We have so far used the prevailing term structure to price a bond or compute the net present value of a project, assuming the spot-rate curve is given.
- In practice, spot rates of interest are not directly observable in the market, although they can be estimated from the observed bond prices.
- We now discuss the estimation of the spot rates of interest, which are assumed to be compounded over discrete time intervals.
- We consider the estimation of the spot rates of interest convertible semiannually using a series of semiannual coupon bonds.

- The simplest way to estimate the spot-rate curve is by the **bootstrap method**. This method requires the bond-price data to follow a certain format.
- In particular, we assume that the coupon-payment dates of the bonds are synchronized and spaced out 6 months apart. The following example illustrates the use of the bootstrap method.

Example 7.7: Table 7.3 summarizes a series of semiannual coupon bonds with different time to maturity, coupon rate of interest r (in percent per annum) and price per 100 face value. Using the given bond data, estimate the spot rate of interest i_t^S over t years, for $t = 0.5, 1, \dots, 6$.

Table 7.3: Bond price data

Maturity (yrs)	Coupon rate r (%)	Price per 100 face value
0.5	0.0	98.41
1.0	4.0	100.79
1.5	3.8	100.95
2.0	4.5	102.66
2.5	2.5	98.53
3.0	5.0	105.30
3.5	3.6	101.38
4.0	3.2	99.83
4.5	4.0	102.83
5.0	3.0	98.17
5.5	3.5	100.11
6.0	3.6	100.24

Solution: We first compute the spot rate of interest for payments due in half-year, which can be obtained from the first bond. Equating the bond price to the present value of the redemption value (there is no coupon), we have

$$98.41 = \frac{100}{1 + \frac{i_{0.5}^S}{2}},$$

which implies

$$i_{0.5}^S = 2 \times \left[\frac{100}{98.41} - 1 \right] = 2 \times 0.016157 = 3.231\%.$$

For the second bond, there are two cash flows. As a coupon is paid at time 0.5 year and the spot rate of interest of which has been computed,

we obtain the following equation of value for the bond:

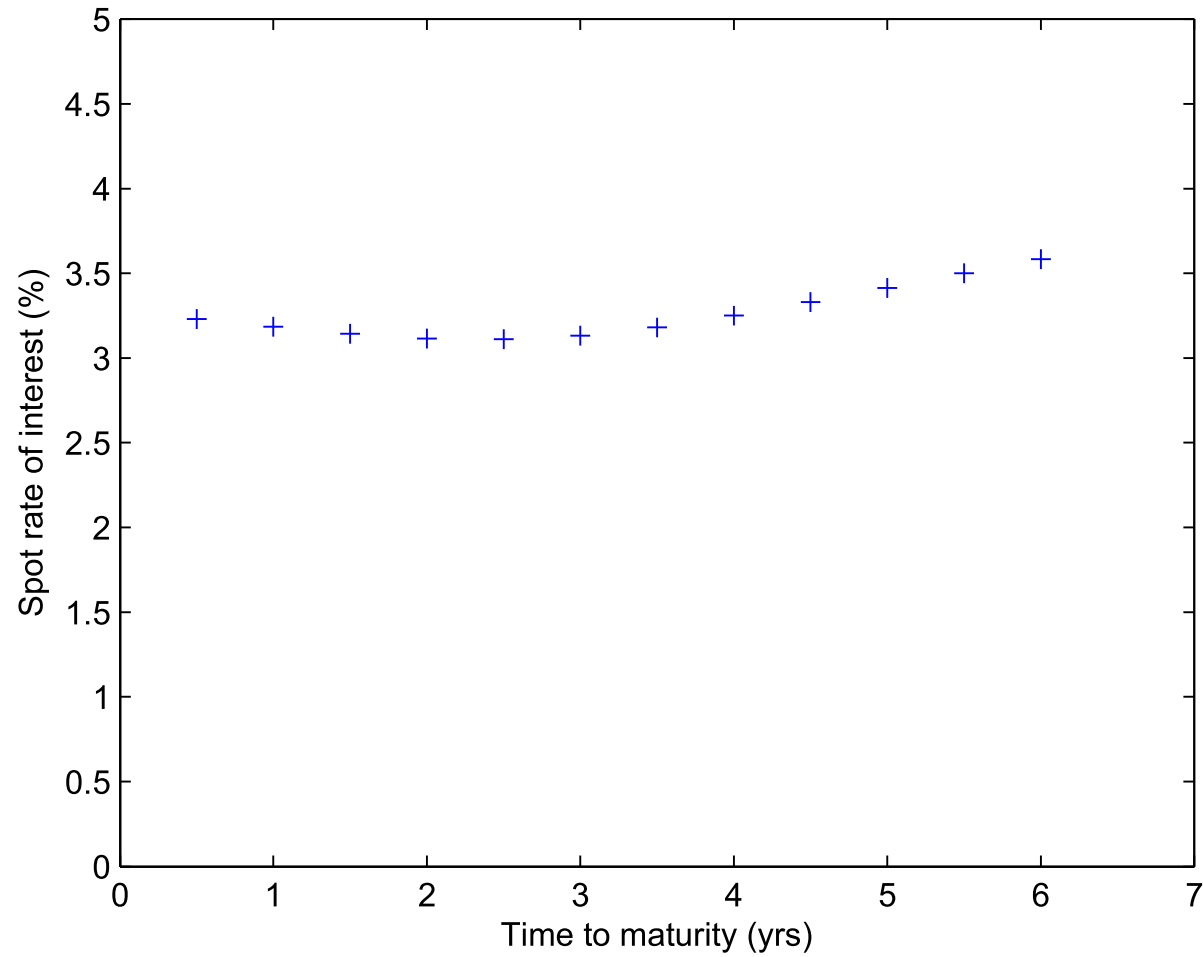
$$100.79 = \frac{2}{1.016157} + \frac{102}{\left(1 + \frac{i_1^S}{2}\right)^2},$$

from which we obtain $i_1^S = 3.191\%$. Similarly, for the third bond the equation of value involves $i_{0.5}^S$, i_1^S and $i_{1.5}^S$, of which only $i_{1.5}^S$ is unknown and can be solved from the equation. Thus, for the bond with k semiannual coupons, price P_k and coupon rate r_k per half-year, the equation of value is

$$P_k = 100 r_k \sum_{j=1}^k \frac{1}{\left(1 + \frac{i_j^S}{2}\right)^j} + \frac{100}{\left(1 + \frac{i_k^S}{2}\right)^k}.$$

These equations can be used to solve sequentially for the spot rates. Figure 7.2 plots the spot-rate curve for maturity of up to 6 years.

Figure 7.2: Estimated spot-rate curve of Example 7.7



□

- Although the bootstrap method is simple to use, there are some data requirements that seriously limit its applicability:
 - the data set of bonds must have synchronized coupon-payment dates
 - there should not be any gap in the series of bonds in the time to maturity
- We now consider the **least squares method**, which is less demanding in the data requirement.
- We assume a set of m bonds for which the coupon-payment dates are synchronized, and denote the prices of these bonds per 100 face value by P_j , their coupon rate by r_j per half-year and their time to maturity by n_j half-years, for $j = 1, \dots, m$.

- We denote

$$v_h = \frac{1}{\left(1 + \frac{iS}{2}\right)^h},$$

which is the discount factor for payments due in h half-years. Thus, the equation of value for the j th bond is

$$P_j = 100 r_j \sum_{h=1}^{n_j} v_h + 100v_{n_j}.$$

- If the last payment (redemption plus coupon) of the bonds occur in M periods (i.e., M is the maximum of all n_j for $j = 1, \dots, m$), the pricing equations of the bonds can be written as

$$P_j = C_{j1}v_1 + C_{j2}v_2 + \dots + C_{jM}v_M, \quad j = 1, \dots, m, \quad (7.10)$$

in which C_{jh} are known cash-flow amounts and v_h are the unknown discount factors.

- Thus, in equation (7.10) we have a **multiple linear regression model** with M unknown coefficients v_1, \dots, v_M and m observations (P_j is the **dependent variable** and C_{j1}, \dots, C_{jM} are the **independent variables**, for $j = 1, \dots, m$).
- We can solve for the values of v_h using the least squares method, and subsequently obtain the values of i_h^S . □

7.6 Continuously Compounded Yield Curve

- We now introduce the estimation of the term structure assuming that interest is credited based on continuous compounding.
- We denote the current time by 0, and use i_t^S to denote the continuously compounded spot rate of interest for payments due at time t .
- As shown in equation (3.27), if we consider the limiting value of the forward rate of interest $i_{t,\tau}^F$ for τ approaching zero, we obtain the instantaneous forward rate, which is equal to the force of interest, i.e.,

$$\lim_{\tau \rightarrow 0} i_{t,\tau}^F = \delta(t).$$

- An important method to construct the yield curve based on the instantaneous forward rate is due to Fama and Bliss (1987).
- If we denote the current price of a zero-coupon bond with unit face value maturing at time t by $P(t)$, we have

$$P(t) = v(t) = \frac{1}{a(t)} = \exp[-t i_t^S], \quad (7.11)$$

and

$$\delta(t) = \frac{a'(t)}{a(t)} = -\frac{v'(t)}{v(t)} = -\frac{P'(t)}{P(t)}. \quad (7.12)$$

- If we had a continuously observable sequence of zero-coupon bond prices $P(t)$ maturing at time $t > 0$, the problem of recovering the spot rates of interest is straightforward using equation (7.11).
- In practice, this sequence is not available.

- The Fama-Bliss method focuses on the estimation of the instantaneous forward rates, from which the spot-rate curve can be computed.

- Recall that

$$a(t) = \exp \left[\int_0^t \delta(u) du \right]. \quad (7.13)$$

- From (7.11) and (7.13), we conclude that

$$\exp [t i_t^S] = \exp \left[\int_0^t \delta(u) du \right],$$

so that

$$i_t^S = \frac{1}{t} \int_0^t \delta(u) du, \quad (7.14)$$

which says that the continuously compounded spot rate of interest is an *equally-weighted average* of the force of interest.

- Thus, if we have a sequence of estimates of the instantaneous forward rates, we can compute the spot-rate curve using (7.14).
- To apply the Fama-Bliss method, we assume that we have a sequence of bonds with possibly irregularly spaced maturity dates. We assume that the force of interest between two successive maturity dates is *constant*.
- Making use of the equations of value for the bonds sequentially in increasing order of the maturity, we are able to compute the force of interest over the period of the data. The spot rates of interest can then be calculated using equation (7.14).
- Although this yield curve may not be smooth, it can be fine tuned using some spline smoothing method.

Example 7.8: You are given the bond data in Table 7.4. The j th bond matures at time t_j with current price P_j , for $j = 1, \dots, 4$ and $0 < t_1 < t_2 < t_3 < t_4$. Bonds 1 and 2 have no coupons, while Bond 3 has one coupon at time t_{31}^* , with $t_1 < t_{31}^* < t_2$. Bond 4 has two coupons with coupon dates t_{41}^* and t_{42}^* , where $t_2 < t_{41}^* < t_3$ and $t_3 < t_{42}^* < t_4$.

Table 7.4: Bond data for Example 7.8

Bond	Coupon per face value	Coupon date	Maturity	Bond price
1	0	—	t_1	P_1
2	0	—	t_2	P_2
3	C_3	$t_{31}^*, t_1 < t_{31}^* < t_2$	t_3	P_3
4	C_4	$t_{41}^*, t_2 < t_{41}^* < t_3$	t_4	P_4
	C_4	$t_{42}^*, t_3 < t_{42}^* < t_4$		

It is assumed that the force of interest $\delta(t)$ follows a step function taking constant values between successive maturities, i.e., $\delta(t) = \delta_i$ for $t_{i-1} \leq t < t_i$, $i = 1, \dots, 4$ with $t_0 = 0$. Estimate $\delta(t)$ and compute the spot-rate curve for maturity up to time t_4 .

Solution: As Bond 1 is a zero-coupon bond, its equation of value is

$$P_1 = \exp \left[-t_1 i_{t_1}^S \right] = \exp \left[- \int_0^{t_1} \delta(u) du \right] = \exp \left[-\delta_1 t_1 \right],$$

so that

$$\delta_1 = -\frac{1}{t_1} \ln P_1,$$

which applies to the interval $(0, t_1)$. Now we turn to Bond 2 (which is again a zero-coupon bond) and write down its equation of value as

$$P_2 = \exp \left[- \int_0^{t_2} \delta(u) du \right] = \exp \left[-\delta_1 t_1 - \delta_2 (t_2 - t_1) \right].$$

Solving the above equation for δ_2 , we obtain

$$\begin{aligned}\delta_2 &= -\frac{1}{t_2 - t_1} [\ln P_2 + \delta_1 t_1] \\ &= -\frac{1}{t_2 - t_1} \ln \left[\frac{P_2}{P_1} \right].\end{aligned}$$

For Bond 3, there is a coupon payment of amount C_3 at time t_{31}^* , with $t_1 < t_{31}^* < t_2$. Thus, the equation of value for Bond 3 is

$$\begin{aligned}P_3 &= \exp \left[-\int_0^{t_3} \delta(u) du \right] + C_3 \exp \left[-\int_0^{t_{31}^*} \delta(u) du \right] \\ &= \exp [-\delta_1 t_1 - \delta_2 (t_2 - t_1) - \delta_3 (t_3 - t_2)] + C_3 \exp [-\delta_1 t_1 - \delta_2 (t_{31}^* - t_1)],\end{aligned}$$

from which we obtain

$$\begin{aligned}\delta_3 &= -\frac{1}{t_3 - t_2} \ln \left[\frac{P_3 - C_3 \exp[-\delta_1 t_1 - \delta_2(t_{31}^* - t_1)]}{P_2} \right] \\ &= -\frac{1}{t_3 - t_2} \ln \left[\frac{P_3 - P_1 C_3 \exp[-\delta_2(t_{31}^* - t_1)]}{P_2} \right].\end{aligned}$$

Going through a similar argument, we can write down the equation of value for Bond 4 as

$$P_4 = P_3^* \exp[-\delta_4(t_4 - t_3)] + P_2 C_4 \exp[-\delta_3(t_{41}^* - t_2)] + P_3^* C_4 \exp[-\delta_4(t_{42}^* - t_3)],$$

where

$$P_3^* = \exp[-\delta_1 t_1 - \delta_2(t_2 - t_1) - \delta_3(t_3 - t_2)],$$

so that

$$\exp[-\delta_4(t_4 - t_3)] + C_4 \exp[-\delta_4(t_{42}^* - t_3)] = \frac{P_4 - P_2 C_4 \exp[-\delta_3(t_{41}^* - t_2)]}{P_3^*}.$$

Thus, the right-hand side of the above equation can be computed, while the left-hand side contains the unknown quantity δ_4 . The equation can be solved numerically for the force of interest δ_4 in the period (t_3, t_4) . Finally, using (7.14), we obtain the continuously compounded spot interest rate as

$$i_t^S = \begin{cases} \delta_1, & \text{for } 0 < t < t_1, \\ \frac{\delta_1 t_1 + \delta_2(t - t_1)}{t}, & \text{for } t_1 \leq t < t_2, \\ \frac{\delta_1 t_1 + \delta_2(t_2 - t_1) + \delta_3(t - t_2)}{t}, & \text{for } t_2 \leq t < t_3, \\ \frac{\delta_1 t_1 + \delta_2(t_2 - t_1) + \delta_3(t_3 - t_2) + \delta_4(t - t_3)}{t}, & \text{for } t_3 \leq t < t_4. \end{cases}$$

□

7.7 Term Structure Models

Figure 7.3: Historical UK spot-rate curves

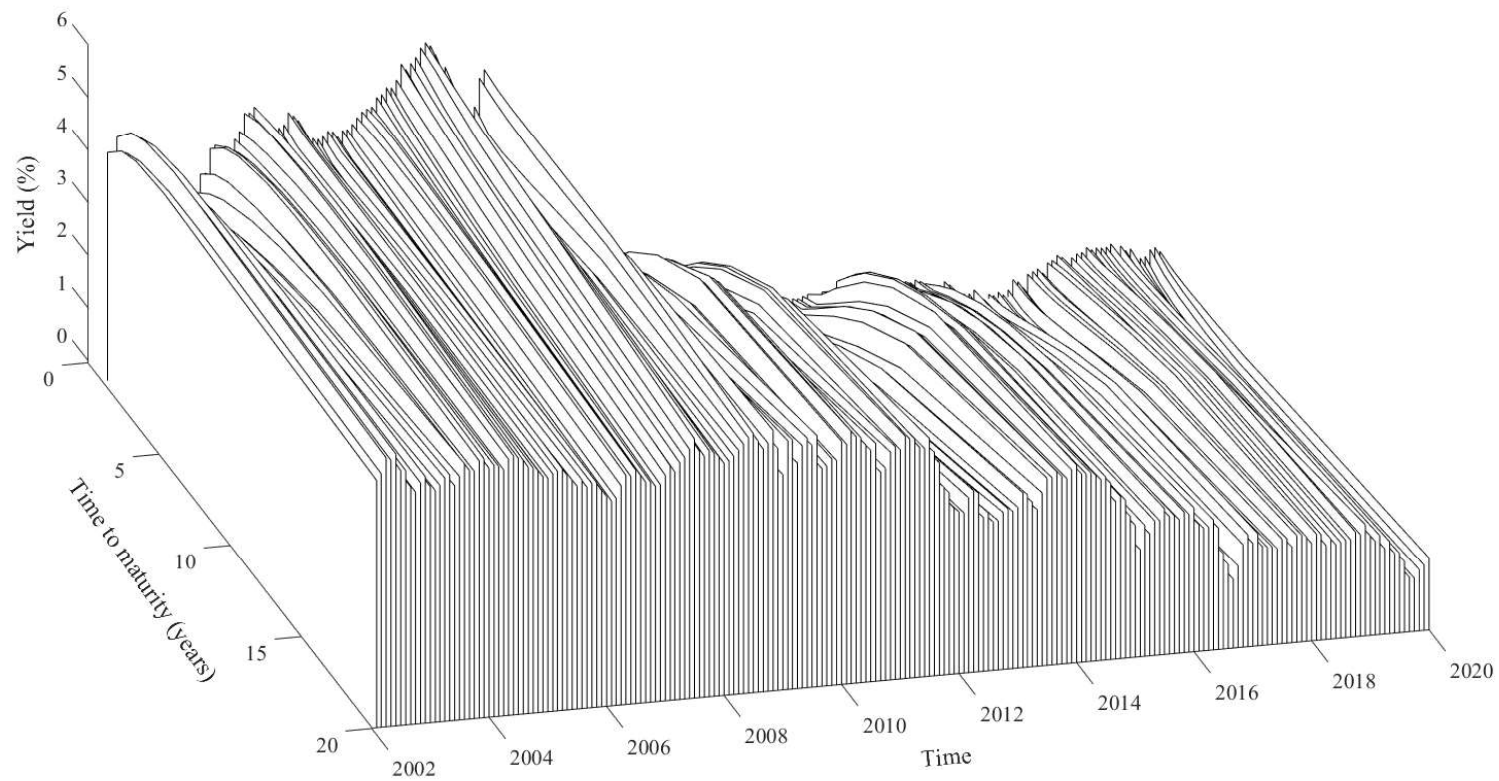
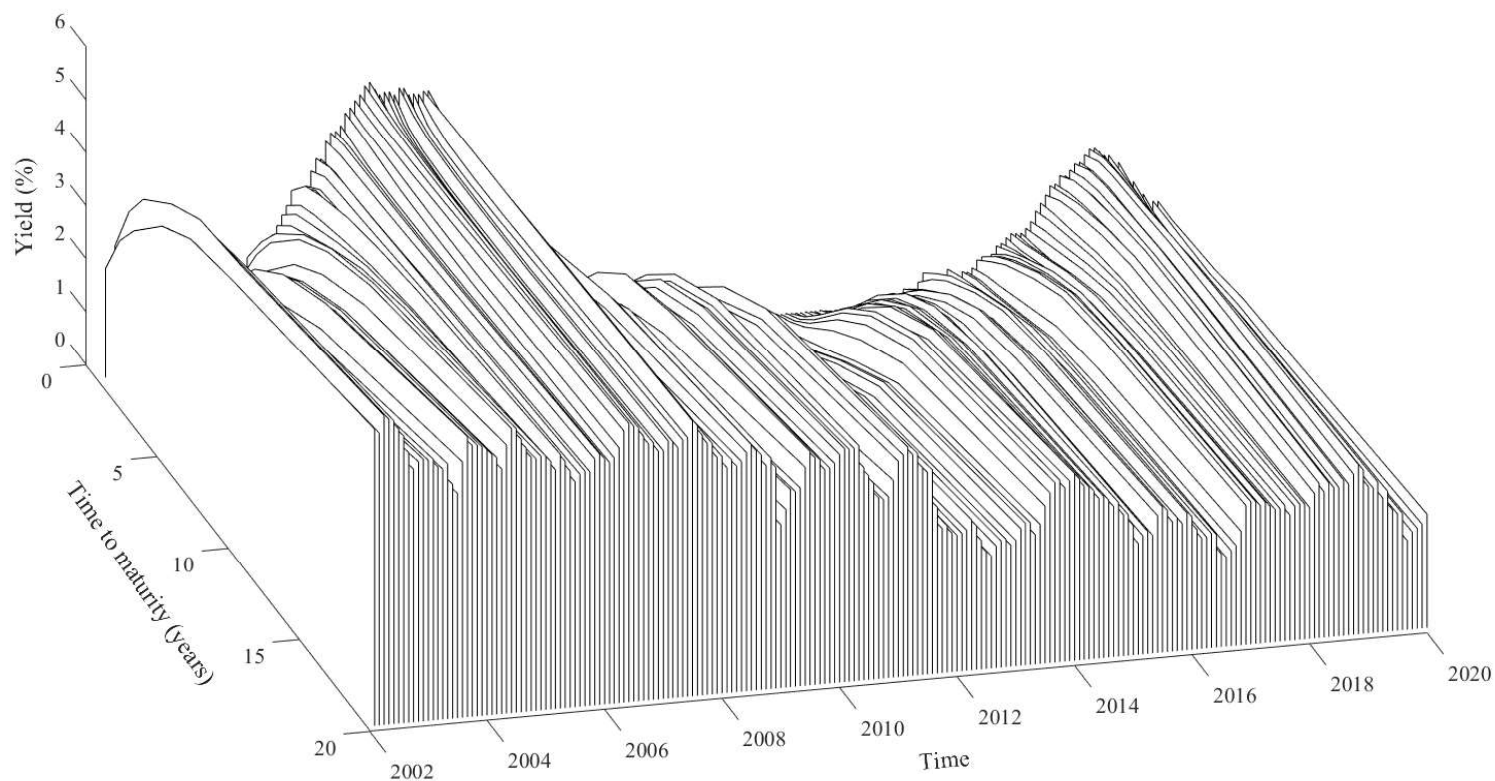


Figure 7.4: Historical US spot-rate curves



- Empirically the yield curve can take various shapes.
- It will be interesting to examine the factors determining the shapes

of the yield curve, how the yield curve evolves over time, and whether the shape of the yield curve has any implications for the real economy such as the business cycle.

- There are several approaches in defining the term-structure models. We shall examine the theories of the term structure in terms of the 1-period holding-period yield introduced in Section 7.4.
- For simplicity of exposition, we assume investments in zero-coupon bonds and relate different measures of returns to the bond prices.
- We denote $i_H^{(n)}$ as the 1-period holding-period (from time 0 to 1) yield of a bond maturing at time n . The current time is 0 and the price of a bond at time t with remaining n periods to mature is denoted by $P_t(n)$.

- We introduce the new notation ${}_t i_\tau^S$ to denote the spot rate of interest at time t for payments due τ periods from t (i.e., due at time $t + \tau$).
- Thus, for $t > 0$, ${}_t i_\tau^S$ is a *random variable* (at time 0).
- Suppose an investor purchased a n -period bond at time 0 and held it for 1 period, the bond price at time 1 is $P_1(n - 1)$, so that the 1-period holding-period yield from time 0 to 1 is

$$i_H^{(n)} = \frac{P_1(n - 1) - P_0(n)}{P_0(n)}. \quad (7.15)$$

- Note that $P_1(n - 1)$ is a random variable at time 0, and so is $i_H^{(n)}$. Furthermore, ${}_t i_\tau^S$ is given by

$${}_t i_\tau^S = \frac{1 - P_t(\tau)}{P_t(\tau)} = \frac{1}{P_t(\tau)} - 1. \quad (7.16)$$

- If we assume that all market participants are **risk neutral** (they neither avoid nor love risks) and that they have no preference for the maturities of the investments, then their only criterion for the selection of an investment is its *expected return*.
- This assumption leads to the **pure expectations hypothesis**, which states that the *expected* 1-period holding-period yields for bonds of all maturities are equal, and thus equal to the 1-period spot rate of interest (which is the 1-period holding-period yield of a bond maturing at time 1).
- If we denote $E[i_H^{(n)}]$ as the expected value of $i_H^{(n)}$ at time 0, the pure expectations hypothesis states that

$$E[i_H^{(n)}] = i_1^S, \quad \text{for } n = 1, \dots. \quad (7.17)$$

- However, while the 1-period holding-period yield for a 1-year zero-coupon bond is known at time 0 (i.e., i_1^S), it is unknown for bonds with longer maturities. Thus, the bonds with longer maturities involve uncertainty and may have higher risks.
- Investors may demand higher expected return for bonds with maturities longer than a year, which is called the **risk premium**.
- If the risk premium is constant for bonds of all maturities, we have the **expectations (or constant premium) hypothesis**, which states that

$$\mathbb{E}[i_H^{(n)}] = i_1^S + \varrho, \quad \text{for } n = 2, \dots, \quad (7.18)$$

where ϱ is a positive constant independent of n .

- However, if investors prefer short-maturity to long-maturity assets due to their better liquidity, then the expected return for bonds with

longer maturities must be higher to compensate the investors.

- This leads to the **liquidity premium hypothesis**, which states that

$$\mathbb{E}[i_H^{(n)}] = i_1^S + \varrho^{(n)}, \quad \text{for } n = 2, \dots, \quad (7.19)$$

where the risk premium for a n -period bond $\varrho^{(n)}$ increases with n , so that $\varrho^{(2)} \leq \varrho^{(3)} \leq \dots$.

- Some theorists argue that investors do not necessarily prefer short-maturity assets to long-maturity assets. Thus, the risk premiums of bonds with different maturities $\varrho^{(n)}$ may not be a monotonic function of n , but may depend on other covariates $w^{(n)}$, which are possibly time varying.
- This is called the **market segmentation hypothesis**, for which

we have

$$\mathbb{E}[i_H^{(n)}] = i_1^S + \varrho(w^{(n)}), \quad \text{for } n = 2, \dots, \quad (7.20)$$

so that the risk premium $\varrho(\cdot)$ is a function of the specific asset market.

- Finally, the **preferred habitat hypothesis** states that institutions do not have a rigid targeted maturity class of assets to invest, but will be influenced by the returns expected of assets with different maturities. Thus, bonds with similar maturities will be close substitutes of each other.
- We now examine the implications of the prevailing term structure for the future movements of interest rates under the aforementioned term-structure models.
- Consider the 1-period holding-period yield of a 2-period bond, $i_H^{(2)}$,

which is given by

$$i_H^{(2)} = \frac{P_1(1) - P_0(2)}{P_0(2)} = \frac{P_1(1)}{P_0(2)} - 1. \quad (7.21)$$

- If we consider the spot rate of interest over the 2-period horizon, i_2^S , we have

$$1 + i_2^S = \frac{1}{P_0(2)} = \frac{P_1(1)}{P_0(2)} \times \frac{1}{P_1(1)} = \left(1 + i_H^{(2)}\right) \left(1 + {}_1i_1^S\right). \quad (7.22)$$

- Using equation (3.4) to rewrite the left-hand side of the above equation, we have

$$\left(1 + i_H^{(2)}\right) \left(1 + {}_1i_1^S\right) = \left(1 + i_1^S\right) \left(1 + i_2^F\right). \quad (7.23)$$

- As rates of interest are small, to the first-order approximation the above equation can be written as

$$i_H^{(2)} + {}_1i_1^S = i_1^S + i_2^F. \quad (7.24)$$

- Now we take expectations on both sides of the equation. If we adopt the pure expectations hypothesis, we have $E[i_H^{(n)}] = i_1^S$, so that equation (7.24) implies

$$E[{}_1i_1^S] = i_2^F, \quad (7.25)$$

which says that the expected value of the *future* 1-period spot rate is equal to the prevailing 1-period forward rate for that period.

- This statement is called the **unbiased expectations hypothesis**, which is itself implied by the pure expectations hypothesis. Note that equation (7.25) can be generalized to

$$E[{}_t i_\tau^S] = i_{t,\tau}^F, \quad \text{for } 0 < t, \tau, \quad (7.26)$$

so that the prevailing forward rates have important implications for the expected future values of the spot rates at any time t over any horizon τ .

- We have seen that if the term structure is upward sloping, the forward rate of interest is higher than the spot rate of interest.
- Thus, under the pure expectations hypothesis, an upward sloping yield curve implies that the future spot rate is *expected* to be higher than the current spot rate. However, if long-term bonds command a risk premium, an upward sloping yield curve may not imply that the spot rate is expected to rise.