Financial Mathematics for Actuaries (Third Edition)

Chapter 5

Loans and Costs of Borrowing

Learning Objectives

1. Loan balance: prospective method and retrospective method

2. Amortization schedule

3. Varying installments and varying interest rates

5.1 Loan Balance: Prospective and Retrospective Methods

- Consider a loan with a fixed term of maturity, to be redeemed by a series of repayments.

- If the repayments prior to maturity are only to offset the interests, the loan is called an **interest-only loan**.

- If the repayments include both payment of interest and partial redemption of the principal, the loan is called a **repayment loan**.

- We consider two approaches to compute the balance of the loan: the **prospective method** and the **retrospective method**.

- The prospective method is forward looking. It calculates the loan balance as the present value of all future payments to be made.

- The retrospective method is backward looking. It calculates the loan balance as the accumulated value of the loan at the time of evaluation minus the accumulated value of all installments paid up to the time of evaluation.

- Let the loan amount be L, and the rate of interest per payment period be i. If the loan is to be paid back in n installments of an annuity-immediate, the installment amount A is given by

$$A = \frac{L}{a_{\overline{n}|i}}. \tag{5.1}$$

- We also denote $L = B_0$, which is the loan balance at time 0.

- Immediately after the mth payment has been made the loan is redeemable by a $n-m$ annuity-immediate. We denote the loan balance after the mth installment by B_m, which is

$$\begin{aligned} B_m &= A\, a_{\overline{n-m}|i} \\ &= \frac{L\, a_{\overline{n-m}|i}}{a_{\overline{n}|i}}. \end{aligned} \tag{5.2}$$

- To use the retrospective method, we first calculate the accumulated loan amount at time m, which is $L(1+i)^m$.

- The accumulated value of the installments is $As_{\overline{m}|i}$.

- Thus, the loan balance can also be written as

$$L(1+i)^m - As_{\overline{m}|i}. \tag{5.3}$$

- To show that the two methods are equivalent, we simplify (5.3).

Thus, we have

$$
\begin{aligned}
Aa_{\overline{n}|i}(1+i)^m - As_{\overline{m}|i} &= A\left[\frac{1-v^n}{i}(1+i)^m - \frac{(1+i)^m - 1}{i}\right] \\
&= A\left[\frac{1-v^{n-m}}{i}\right] \\
&= A\, a_{\overline{n-m}|i}.
\end{aligned}
$$

Example 5.1: A housing loan of $400,000 was to be repaid over 20 years by monthly installments of an annuity-immediate at the nominal rate of 5% per year. After the 24th payment was made, the bank increased the interest rate to 5.5%. If the lender was required to repay the loan within the same period, how much would be the increase in the monthly installment. If the installment remained unchanged, how much longer would it take to pay back the loan?

Solution: We first demonstrate the use of the prospective and retrospective methods for the calculation of the loan balance after the 24th payment. From (5.1), the amount of the monthly installment is

$$\begin{aligned} A &= \frac{400{,}000}{a_{\overline{240}\rceil 0.05/12}} \\ &= \frac{400{,}000}{151.525} \\ &= \$2{,}639.82. \end{aligned}$$

By the prospective method, after the 24th payment the loan would be redeemed with a 216-payment annuity-immediate so that the balance is

$$\begin{aligned} A\, a_{\overline{216}\rceil 0.05/12} &= 2{,}639.82 \times 142.241 \\ &= \$375{,}490. \end{aligned}$$

By the retrospective method, the balance is

$$400{,}000\left(1+\frac{0.05}{12}\right)^{24} - 2{,}639\, s_{\overline{24}|0.05/12} = 441{,}976.53 - 2{,}639.82 \times 25.186$$
$$= \$375{,}490.$$

Hence, the two methods give the same answer. After the increase in the rate of interest, if the loan is to be repaid within the same period, the revised monthly installment is

$$\frac{375{,}490}{a_{\overline{216}|0.055/12}} = \frac{375{,}490}{136.927}$$
$$= \$2{,}742.26,$$

so that the increase in installment is \$102.44. Let m be the remaining number of installments if the amount of installment remained unchanged.

Thus,

$$a_{\overline{m}|0.055/12} = a_{\overline{216}|0.05/12} = 142.241,$$

from which we solve for $m = 230.88 \approx 231$. Thus, it takes 15 months more to pay back the loan. □

Example 5.2: A housing loan is to be repaid with a 15-year monthly annuity-immediate of \$2,000 at a nominal rate of 6% per year. After 20 payments, the borrower requests for the installments to be stopped for 12 months. Calculate the revised installment when the borrower starts to pay back again, so that the loan period remains unchanged. What is the difference in the interest paid due to the temporary stoppage of installments?

Solution: The loan is to be repaid over $15 \times 12 = 180$ payments. After 20 payments, the loan still has 160 installments to be paid. Using the prospective method, the balance of the loan after 20 payments is

$$2{,}000 \times a_{\overline{160}|0.005} = \$219{,}910.$$

Note that if we calculate the loan balance using the retrospective method, we need to compute the original loan amount. The full calculation using the retrospective method is

$$\begin{aligned} 2{,}000\, a_{\overline{180}|0.005}\, (1.005)^{20} - 2{,}000\, s_{\overline{20}|0.005} &= 2{,}000\, (130.934 - 20.979) \\ &= \$219{,}910. \end{aligned}$$

Due to the delay in payments, the loan balance 12 months after the 20th payment is

$$219{,}910\, (1.005)^{12} = \$233{,}473,$$

which has to be repaid with a 148-payment annuity-immediate. Hence, the revised installment is

$$\frac{233{,}473}{a_{\overline{148}|0.005}} = \frac{233{,}473}{104.401}$$
$$= \$2{,}236.31.$$

The difference in the interest paid is

$$2{,}236.31 \times 148 - 2{,}000 \times 160 = \$10{,}973.$$

□

Example 5.3: A man borrows a housing loan of \$500,000 from Bank A to be repaid by monthly installments over 20 years at nominal rate of interest of 4% per year. After 24 installments Bank B offers the man a loan at rate of interest of 3.5% to be repaid over the same period. However, if the man wants to re-finance the loan he has to pay Bank A a penalty equal to 1.5% of the outstanding balance. If there are no other re-financing costs, should the man re-finance the loan?

Solution: The monthly installment paid to Bank A is

$$\frac{500{,}000}{a_{\overline{240}\rceil 0.04/12}} = \frac{500{,}000}{165.02} = \$3{,}029.94.$$

The outstanding balance after paying the 24th installment is

$$3{,}029.94\, a_{\overline{216}\rceil 0.04/12} = 3{,}029.94 \times 153.80 = \$466{,}004.$$

If the man re-finances with Bank B, he needs to borrow

$$466{,}004 \times 1.015 = \$472{,}994,$$

so that the monthly installment is

$$\frac{472{,}994}{a_{\overline{216}|0.035/12}} = \frac{472{,}994}{160.09}$$
$$= \$2{,}954.56.$$

As this is less than the installments of \$3,029.94 he pays to Bank A, he should re-finance. □

5.2 Amortization

- If a loan is repaid by the amortization method, each installment is first used to offset the interest incurred since the last payment.
- The remaining part of the installment is then used to reduce the principal.
- Consider a loan to be repaid by a n-payment unit annuity-immediate. The principal is $a_{\overline{n}|i}$ and the interest incurred in the first payment period is $i\,a_{\overline{n}|i} = 1 - v^n$.
- Thus, out of the unit payment, $1 - v^n$ goes to paying the interest and the principal is reduced by the amount v^n.

- The principal is then reduced to

$$\begin{aligned} a_{\overline{n}\rceil i} - v^n &= \frac{1 - v^n}{i} - v^n \\ &= \frac{1 - v^n(1+i)}{i} \\ &= \frac{1 - v^{n-1}}{i} \\ &= a_{\overline{n-1}\rceil i}. \end{aligned}$$

- Making use of this result we can construct an **amortization schedule** that separates each installment into the interest and principal components.

Table 5.1: Amortization of a loan of $a_{\overline{n}|i}$ by a n-payment unit annuity-immediate at effective interest rate of i per period

Time	Installment	Interest payment	Principal payment	Outstanding balance
0				$a_{\overline{n}\rceil}$
1	1	$i\,a_{\overline{n}\rceil} = 1 - v^n$	v^n	$a_{\overline{n}\rceil} - v^n = a_{\overline{n-1}\rceil}$
·	·	·	·	·
·	·	·	·	·
t	1	$i\,a_{\overline{n-t+1}\rceil} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}\rceil} - v^{n-t+1} = a_{\overline{n-t}\rceil}$
·	·	·	·	·
·	·	·	·	·
n	1	$i\,a_{\overline{1}\rceil} = 1 - v$	v	$a_{\overline{1}\rceil} - v = 0$
Total	n	$n - a_{\overline{n}\rceil}$	$a_{\overline{n}\rceil}$	

Example 5.4: Construct an amortization schedule of a loan of \$5,000 to be repaid over 6 years with a 6-payment annuity-immediate at effective rate of interest of 6% per year.

Solution: The annual payment is

$$\frac{5{,}000}{a_{\overline{6}|0.06}} = \frac{5{,}000}{4.9173}$$
$$= \$1{,}016.81.$$

Using Table 5.1 with installments of 1,016.81 in place of 1, we obtain the amortization schedule in Table 5.2.

Table 5.2: Amortization of a loan of \$5,000 by a 6-payment annuity-immediate at effective interest rate of 6% per year

Year	Installment	Interest payment	Principal payment	Outstanding balance
0				5,000.00
1	1,016.81	300.00[a]	716.81[b]	4,283.19[c]
2	1,016.81	256.99[d]	759.82	3,523.37
3	1,016.81	211.40	805.41	2,717.96
4	1,016.81	163.08	853.73	1,864.23
5	1,016.81	111.85	904.96	959.26
6	1,016.81	57.55	959.26	0.00
Total	6,100.86	1,100.86	5,000.00	

The details of the computation in Table 5.2 are given as follows:

[a] Interest incurred in the first period: $5{,}000 \times 0.06 = \$300$.

[b] Principal reduction with the first installment: $1{,}016.81 - 300 = \$716.81$.

[c] Outstanding balance at the end of the first year: $5{,}000 - 716.81 =$ \$4,283.19.

[d] Interest incurred in the second period: $4{,}283.19 \times 0.06 =$ \$256.99.

The rest of the computations follows similarly. □

5.4 Varying Installments and Varying Interest Rates

- Installments to repay a loan need not be level. Suppose a loan over n periods at rate of interest i is to be paid back by a varying n-payment annuity-immediate $A_1, \cdots, A_n$. The loan amount L is given by

$$L = \sum_{t=1}^{n} A_t v^t. \tag{5.10}$$

- The balance of the loan B_{k-1} after the $(k-1)$th payment is

$$B_{k-1} = \sum_{t=k}^{n} A_t v^{t-k+1}.$$

- Thus, the interest in the kth payment is iB_{k-1} and the principal is reduced by the amount $A_k - iB_{k-1}$.

Example 5.9: A 20-year loan is to be repaid by installments of \$100 at the end of year 1, \$200 at the end of year 2, and so on, increasing by \$100 each year, up to \$600 at the end of the sixth year. From then onwards, a level installment of \$600 will be paid. The rate of interest charged is 6%. Calculate the loan amount. What is the interest and the principal reduction in the 3rd and the 12th payments?

Solution: The time diagram of the repayment is shown in Figure 5.3. The loan amount L is

$$\begin{aligned}100(Ia)_{\overline{6}|} + 600v^6 a_{\overline{14}|} &= 100 \times \left[\frac{5.2124 - 6(1.06)^{-6}}{0.06}\right] + 3{,}931.56 \\ &= \$5{,}569.22.\end{aligned}$$

To calculate the balance after the second installment, we use the retrospective method to obtain

$$B_2 = 5{,}569.22(1.06)^2 - 100(1.06) - 200 = \$5{,}951.57.$$

Note that the loan amount has actually increased, as the initial payments are not sufficient to cover the interest. This is the case of a *negative amortization*. The interest in the third payment is

$$0.06 \times 5{,}951.57 = \$357.09.$$

The third installment of \$300 is not sufficient to pay for the interest, and there is an increase of loan balance by \$57.09 to \$6,008.66.

To split the 12th installment into interest and principal, we first calculate the loan balance after the 11th payment using the prospective method, which is

$$\begin{aligned} B_{11} &= 600a_{\overline{9}|} \\ &= \$4{,}081.02. \end{aligned}$$

Thus, the interest in the 12th payment is $4{,}081.02 \times 0.06 = \$244.86$, so that the principal reduction is $600 - 244.86 = \$355.14$. □

- For the sinking fund method in which the loan charges rate of interest i and the sinking fund credits interest at the rate j, the interest in each installment is iL.

- If the sinking fund contribution at time t is S_t for $t = 1, \cdots, n$, then

$$\sum_{t=1}^{n} S_t(1+j)^{n-t} = L, \tag{5.11}$$

and the tth installment is

$$A_t = S_t + iL.$$

Example 5.10: Suppose a 20-year loan of \$1,000 at rate of interest 6% is to be redeemed by the sinking fund method which credits sinking fund deposits at 5.5%. If the sinking fund deposit increases by 5% in each payment, what is the amount of the sixth installment of this loan? What is the total amount of installments to repay this loan?

Solution: Let S be the first sinking fund deposit. The tth sinking fund deposit is $S(1.05)^{t-1}$ for $t = 1, \cdots, 20$. As the deposits accumulate to \$1,000 in 20 years, we have

$$
\begin{aligned}
1{,}000 &= \sum_{t=1}^{20} S(1.05)^{t-1}(1.055)^{20-t} \\
&= \frac{S(1.055)^{20}}{1.05} \sum_{t=1}^{20} \left(\frac{1.05}{1.055}\right)^{t} \\
&= 2.7788S \sum_{t=1}^{20} (0.9953)^{t} \\
&= 52.91S,
\end{aligned}
$$

from which we solve for $S = \$18.90$. Thus, the 6th installment is

$$1{,}000 \times 0.06 + 18.90(1.05)^{5} = \$84.12.$$

The total amount paid is

$$1{,}000 \times 0.06 \times 20 + 18.90 \sum_{t=0}^{19} (1.05)^t = \$1{,}824.95.$$

For comparison, the total payments using the amortization method is

$$20 \times \frac{1{,}000}{a_{\overline{20}\rceil 0.06}} = \$1{,}743.69.$$

□

Example 5.11: A man borrows a housing loan of \$300,000 from a bank for 20 years, to be paid back by monthly installments. In the first year the bank charges an interest rate of 2.5%, followed by the second year of 3%. From then onwards, the interest rate will be 5%. All rates quoted are nominal per year. The bank calculates installments *as if* the loan is to be repaid over the remaining term at the ongoing rate. Thus, the first year installment assumes rate of interest of 2.5% for 20 years.

(a) Calculate the installments in the first, second and third year.

(b) What are the interests paid in the first and the second year?

Solution: (a) The first-year payment is

$$\frac{300{,}000}{a_{\overline{240}|0.025/12}} = \frac{300{,}000}{188.71}$$
$$= \$1{,}589.71.$$

Using the prospective method, the loan balance after 12 payments is

$$1{,}589.71a_{\overline{228}|0.025/12} = \$288{,}290$$

so that the revised installment is

$$\frac{288{,}290}{a_{\overline{228}|0.03/12}} = \frac{288{,}290}{173.63}$$
$$= \$1{,}660.38.$$

Repeating the same calculations, the balance after 24 payments is $1{,}660.38\, a_{\overline{216}|0.03/12} = \$276{,}858$. The revised installment is

$$\frac{276{,}858}{a_{\overline{216}|0.05/12}} = \$1{,}946.41.$$

For (b), the total amount of payments in the first year is $12 \times 1{,}589.71 = \$19{,}076.52$, and the reduction in principal is $300{,}000 - 288{,}290 = \$11{,}710$. Thus, the interest served is $19{,}076 - 11{,}710 = \$7{,}366$. Likewise, the interest served in the second year is $12 \times 1{,}660.38 - (288{,}290 - 276{,}858) = \$8{,}492.56$.

□