Financial Mathematics for Actuaries (Third Edition)

Chapter 4

Rates of Return

Learning Objectives

- Internal rate of return (yield rate)
- One-period rate of return of a fund: time-weighted rate of return and dollar-weighted (money-weighted) rate of return
- Rate of return over longer periods: geometric mean rate of return and arithmetic mean rate of return
- Portfolio return and return of a short-selling strategy

4.1 Internal Rate of Return

- Consider a project with initial investment C_0. We assume the cash flows occur at regular intervals.

- The project lasts for n years and the future cash flows are denoted by $C_1, \cdots, C_n$.

- We adopt the convention that cash inflows to the project (investments) are positive and cash outflows from the project (withdrawals) are negative.

- We define the **internal rate of return (IRR)** (also called the **yield rate**) as the rate of interest such that the sum of the present values of the cash flows is equated to zero.

- Denoting the internal rate of return by y, we have

$$\sum_{j=0}^{n} \frac{C_j}{(1+y)^j} = 0, \tag{4.1}$$

where j is the time at which the cash flow C_j occurs.

- This equation can also be written as

$$C_0 = -\sum_{j=1}^{n} \frac{C_j}{(1+y)^j}. \tag{4.2}$$

- Thus, the net present value of all future withdrawals (injections are negative withdrawals) evaluated at the IRR is equal to the initial investment.

Example 4.1: A project requires an initial cash outlay of \$2,000 and is expected to generate \$800 at the end of year 1 and \$1,600 at the end of year 2, at which time the project will terminate. Calculate the IRR of the project.

Solution: If we denote $v = 1/(1+y)$, we have, from (4.2)

$$2{,}000 = 800v + 1{,}600v^2,$$

or

$$5 = 2v + 4v^2.$$

Dropping the negative answer from the quadratic equation, we have

$$v = \frac{-2 + \sqrt{4 + 4 \times 4 \times 5}}{2 \times 4} = 0.8956.$$

Thus, $y = (1/v) - 1 = 11.66\%$. Note that $v < 0$ implies $y < -1$, i.e., the loss is larger than 100%, which is precluded from consideration. $\square$

- There is generally no analytic solution for y in (4.1) when $n > 2$, and numerical methods have to be used. The Excel function `IRR` enables us to compute the answer easily. Its usage is described as follows:

Excel function: IRR(**values**,guess)
values = an array of values containing the cash flows
guess = starting value, set to 0.1 if omitted
Output = IRR of cash flows

Example 4.2: An investor pays \$5 million for a 5-year lease of a shopping mall. He will receive \$1.2 million rental income at the end of each year. Calculate the IRR of his investment.

Solution: See Exhibit 4.1. □

- If the cash flows occur more frequently than once a year, such as monthly or quarterly, y computed from (4.1) is the IRR for the payment interval.

- Suppose cash flows occur m times a year, the nominal IRR in annualized term is $m \cdot y$, while the annual effective rate of return is $(1+y)^m - 1$.

Example 4.3: A cash outlay of \$100 generates incomes of \$20 after 4 months and 8 months, and \$80 after 2 years. Calculate the IRR of the investment.

Solution: If we treat one month as the interest conversion period, the equation of value can be written as

$$100 = \frac{20}{(1+y_1)^4} + \frac{20}{(1+y_1)^8} + \frac{80}{(1+y_1)^{24}},$$

where y_1 is the IRR on monthly interval. The nominal rate of return on monthly compounding is $12y_1$. Alternatively, we can use the 4-month interest conversion interval, and the equation of value is

$$100 = \frac{20}{1+y_4} + \frac{20}{(1+y_4)^2} + \frac{80}{(1+y_4)^6},$$

where y_4 is the IRR on 4-month interval. The nominal rate of return on 4-monthly compounding is $3y_4$. The effective annual rate of return is $y = (1+y_1)^{12} - 1 = (1+y_4)^3 - 1$.

The above equations of value have to be solved numerically for y_1 or y_4. We obtain 1.0406% as the IRR per month, namely, y_1. The effective annualized rate of return is then $(1.010406)^{12} - 1 = 13.23\%$. Solving for y_4 with Excel, we obtain the answer 4.2277%. Hence, the annualized effective rate is $(1.042277)^3 - 1 = 13.23\%$, which is equal to the effective rate computed using y_1. $\square$

- When cash flows occur irregularly, we can define y as the annualized rate and express all time of occurrence of cash flows in years.

- Suppose there are $n+1$ cash flows occurring at time 0, t_1, $\cdots$, t_n, with cash amounts C_0, C_1, $\cdots$, C_n. Equation (4.2) is rewritten as

$$C_0 = -\sum_{j=1}^{n} \frac{C_j}{(1+y)^{t_j}}, \tag{4.3}$$

which requires numerical methods for the solution of y.

- We may also use the Excel function `XIRR` to solve for y. Unlike `IRR`, `XIRR` allows the cash flows to occur at *irregular* time intervals. The specification of `XIRR` is as follows:

Excel function: XIRR(values,dates,guess)
values = C_j, an array of values containing the cash flows in (4.3) **dates** = t_j, an array of values containing the timing of the cash flows in (4.3) guess = guess starting value, set to 0.1 if omitted Output = IRR, value of annualized y satisfying (4.3)

Example 4.4: A project requires an outlay of \$2.35 million in return for \$0.8 million after 9 months, \$1 million after 15 months and \$1 million after 2 years. What is the IRR of the project?

Solution: Returns of the project occur at time (in years) 0.75, 1.25 and 2. We solve for v numerically from the following equation using Excel Solver (see Exhibit 4.2)

$$235 = 80v^{0.75} + 100v^{1.25} + 100v^2$$

to obtain $v = 0.879$, so that

$$y = \frac{1}{0.879} - 1 = 13.77\%,$$

which is the effective annualized rate of return of the project. Alternatively, we may use the Excel function `XIRR` as shown in Exhibit 4.3. □

- A project with no subsequent investment apart from the initial capital is called a **simple project**.

- For simple projects, (4.1) has a unique solution with $y > -1$, so that IRR is well defined.

Example 4.5: A project requires an initial outlay of \$8 million, generates returns of \$50 million 1 year later, and requires \$50 million to terminate at the end of year 2. Solve for y in (4.1).

Solution: We are required to solve

$$8 = 50v - 50v^2,$$

which has $v = 0.8$ and 0.2 as solutions. This implies y has multiple solutions of 25% and 400%. $\square$

4.2 One-Period Rate of Return

- We consider methods of calculating the return of a fund over a 1-period interval. The methodology adopted depends on the data available.

- We start with the situation where the exact amounts of fund withdrawals and injections are known, as well as the time of their occurrence.

- Consider a 1-year period with initial fund amount B_0 (equal to C_0). Cash flow of amount C_j occurs at time t_j (in fraction of a year) for $j = 1, \cdots, n$, where $0 < t_1 < \cdots < t_n < 1$.

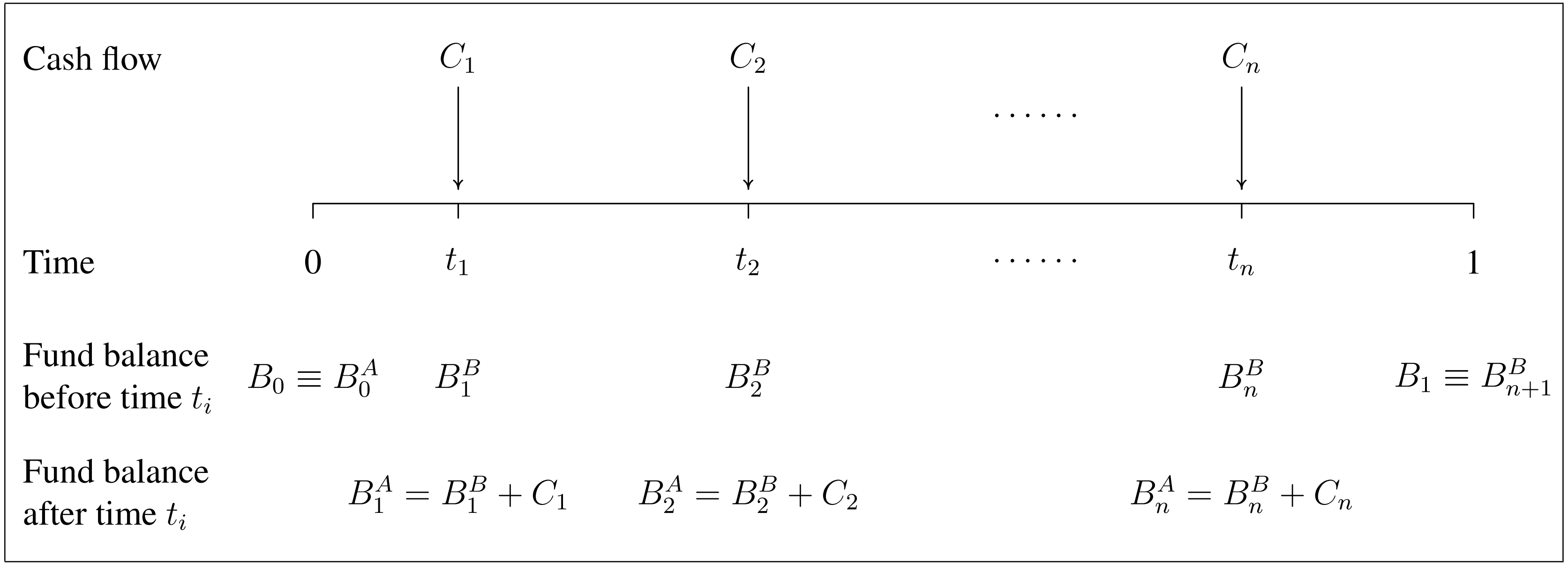
- Note that C_j are usually fund redemptions and new investments, and do not include investment incomes such as dividends and coupon payments.

- Denoting the fund value before and after the transaction at time t_j by B_j^B and B_j^A, respectively, we have $B_j^A = B_j^B + C_j$ for $j = 1, \cdots, n$.

- The difference between B_j^B and B_{j-1}^A, i.e., the balance before the transaction at time t_j and after the transaction at time t_{j-1}, is due to investment incomes, as well as capital gains and losses.

- Let the fund balance at time 1 be B_1, and define $B_{n+1}^B = B_1$ and $B_0^A = B_0$ (this notation will allow us to express the gross return as (4.4) below). Figure 4.1 illustrates the time diagram.

Figure 4.1: Cash flows and fund values in period (0, 1)



- We now introduce two methods to calculate the 1-year rate of return: the **time-weighted rate of return** (TWRR) and the **dollar-weighted rate of return** (DWRR).

- To compute the TWRR we first calculate the return over each subinterval between the occurrences of transactions by comparing the fund balances just before the new transaction to the fund balance just after the last transaction.

- If we denote R_j as the rate of return over the subinterval t_{j-1} to t_j, we have

$$1 + R_j = \frac{B_j^B}{B_{j-1}^A}, \quad \text{for } j = 1, \cdots, n+1. \tag{4.4}$$

- Then TWRR over the year, denoted by R_T, is

$$R_T = \left[\prod_{j=1}^{n+1}(1 + R_j)\right] - 1. \tag{4.5}$$

- The TWRR requires data of the fund balance prior to each withdrawal or injection. In contrast, the DWRR does not require this information. It only uses the information of the amounts of the withdrawals and injections, as well as their time of occurrence.

- In principle, when cash of amount C_j is injected (withdrawn) at time t_j, there is a gain (loss) of capital of amount $C_j(1 - t_j)$ for the remaining period of the year.

- Thus, the *effective capital* of the fund over the 1-year period, denoted by B, is given by

$$B = B_0 + \sum_{j=1}^{n} C_j(1 - t_j).$$

- Denoting $C = \sum_{j=1}^{n} C_j$ as the net injection of cash (withdrawal if negative) over the year and I as the interest income earned over the year, we have $B_1 = B_0 + I + C$, so that

$$I = B_1 - B_0 - C. \tag{4.6}$$

Hence the DWRR over the 1-year period, denoted by R_D, is

$$R_D = \frac{I}{B} = \frac{B_1 - B_0 - C}{B_0 + \sum_{j=1}^{n} C_j(1 - t_j)}. \tag{4.7}$$

Example 4.6: On January 1, a fund was valued at 100k (1k = 1,000). On May 1, the fund increased in value to 112k and 30k of new principal was injected. On November 1, the fund value dropped to 125k, and 42k was withdrawn. At the end of the year, the fund was worth 100k. Calculate the DWRR and the TWRR.

Solution: As $C = 30 - 42 = -12$, there is a net withdrawal. From

(4.6), the interest income earned over the year is

$$I = 100 - 100 - (-12) = 12.$$

Hence, from (4.7), the DWRR is

$$R_D = \frac{12}{100 + \dfrac{2}{3} \times 30 - \dfrac{1}{6} \times 42} = 10.62\%.$$

The fund balance just after the injection on May 1 is $112 + 30 = 142$k, and its value just after the withdrawal on November 1 is $125 - 42 = 83$k. From (4.4), the fund-value relatives over the three subperiods are

$$1 + R_1 = \frac{112}{100} = 1.120,$$

$$1 + R_2 = \frac{125}{142} = 0.880,$$

$$1 + R_3 = \frac{100}{83} = 1.205.$$

Hence, from (4.5), the TWRR is

$$R_T = 1.120 \times 0.880 \times 1.205 - 1 = 18.76\%.$$

$\square$

- TWRR compounds the returns of the fund over subperiods after purging the effects of the timing and amount of cash injections and withdrawals.

- As fund managers have no control over the timing of fund injection and withdrawal, the TWRR appropriately measures the *performance of the fund manager*.

- DWRR is sensitive to the timing and amount of the cash flows.

- If the purpose is to measure the *performance of the fund*, the DWRR is appropriate.

- It allows superior market timing to impact the return of the fund.

- For funds with frequent cash injections and withdrawals, the computation of the TWRR may not be feasible. The difficulty lies in the evaluation of the fund value B_j^B, which requires the fund to be constantly **marked to market**.

- In some situations the exact timing of the cash flows may be difficult to identify.

- In this situation, we may approximate (4.7) by assuming the cash flows to be evenly distributed throughout the 1-year evaluation period.

- Hence, we replace $1-t_j$ by its mean value of 0.5 so that $\sum_{j=1}^{n} C_j(1-t_j) = 0.5C$, and (4.7) can be written as

$$\begin{aligned} R_D &\simeq \frac{I}{B_0 + 0.5C} \\ &= \frac{I}{B_0 + 0.5(B_1 - B_0 - I)} \\ &= \frac{I}{0.5(B_1 + B_0 - I)}. \end{aligned} \tag{4.8}$$

Example 4.7: For the data in Example 4.6, calculate the approximate value of the DWRR using (4.8).

Solution: With $B_0 = B_1 = 100$, and $I = 12$, the approximate R_D is

$$R_D = \frac{12}{0.5(100 + 100 - 12)} = 12.76\%.$$

□

4.3 Rate of Return over Multiple Periods

- We now consider the rate of return of a fund over a m-year period.

- We first consider the case where only annual data of returns are available. Suppose the annual rates of return of the fund have been computed as $R_1, \cdots, R_m$. Note that $-1 \leq R_j < \infty$ for all j.

- The average return of the fund over the m-year period can be calculated as the mean of the sample values. We call this measure the **arithmetic mean rate of return**, denoted by R_A, which is given by

$$R_A = \frac{1}{m} \sum_{j=1}^{m} R_j. \tag{4.9}$$

- An alternative is to use the geometric mean to calculate the average, called the **geometric mean rate of return**, denoted by R_G, which is given by

$$R_G = \left[\prod_{j=1}^{m}(1+R_j)\right]^{\frac{1}{m}} - 1$$
$$= [(1+R_1)(1+R_2)\cdots(1+R_m)]^{\frac{1}{m}} - 1. \qquad (4.10)$$

Example 4.8: The annual rates of return of a bond fund over the last 5 years are (in %) as follows:

$$6.4 \quad 8.9 \quad 2.5 \quad -2.1 \quad 7.2$$

Calculate the arithmetic mean rate of return and the geometric mean rate of return of the fund.

Solution: The arithmetic mean rate of return is

$$\begin{aligned} R_A &= (6.4 + 8.9 + 2.5 - 2.1 + 7.2)/5 \\ &= 22.9/5 \\ &= 4.58\%, \end{aligned}$$

and the geometric mean rate of return is

$$\begin{aligned} R_G &= (1.064 \times 1.089 \times 1.025 \times 0.979 \times 1.072)^{\frac{1}{5}} - 1 \\ &= (1.246)^{\frac{1}{5}} - 1 \\ &= 4.50\%. \end{aligned}$$

$\square$

Example 4.9: The annual rates of return of a stock fund over the last 8 years are (in %) as follows:

$$15.2 \quad 18.7 \quad -6.9 \quad -8.2 \quad 23.2 \quad -3.9 \quad 16.9 \quad 1.8$$

Calculate the arithmetic mean rate of return and the geometric mean rate of return of the fund.

Solution: The arithmetic mean rate of return is

$$R_A = (15.2 + 18.7 + \cdots + 1.8)/8 = 7.10\%$$

and the geometric mean rate of return is

$$R_G = (1.152 \times 1.187 \times \cdots \times 1.018)^{\frac{1}{8}} - 1 = 6.43\%.$$

$\square$

- Given any sample of data, the arithmetic mean is always larger than the geometric mean.

- If the purpose is to measure the return of the fund over the *holding period* of m years, the geometric mean rate of return is the appropriate measure.

- The arithmetic mean rate of return describes the average performance of the fund for *one* year taken randomly from the sample period.

- If there are more data about the history of the fund, alternative measures of the performance of the fund can be used. The methodology of the time-weighted rate of return in Section 4.2 can be extended to beyond 1 period (year).

- Suppose there are n subperiods, with returns denoted by $R_1, \cdots, R_n$, over a period of m years. Then we can measure the m-year return by compounding the returns over each subperiod to form the

TWRR using the formula

$$R_T = \left[\prod_{j=1}^{n}(1+R_j)\right]^{\frac{1}{m}} - 1. \tag{4.11}$$

- We can also compute the return over a m-year period using the IRR. We extend the notations for cash flows in Section 4.2 to the m-year period.

- Suppose cash flows of amount C_j occur at time t_j for $j = 1, \cdots, n$, where $0 < t_1 < \cdots < t_n < m$. Let the fund value at time 0 and time m be B_0 and B_1, respectively. We treat $-B_1$ as the last transaction, i.e., fund withdrawal of amount B_1.

- The rate of return of the fund is calculated as the IRR which equates the discounted values of $B_0, C_1, \cdots, C_n$, and $-B_1$ to zero.

- This is referred to as the DWRR over the m-year period. We denote it as R_D, which solves the following equation

$$B_0 + \sum_{j=1}^{n} \frac{C_j}{(1+R_D)^{t_j}} - \frac{B_1}{(1+R_D)^m} = 0. \tag{4.12}$$

- The example below concerns the returns of a bond fund. When a bond makes the periodic coupon payments, the bond values drop and the coupons are cash amounts to be withdrawn from the fund.

Example 4.10: A bond fund has an initial value of \$20 million. The fund records coupon payments in six-month periods. Coupons received from January 1 through June 30 are regarded as paid on April 1. Likewise, coupons received from July 1 through December 31 are regarded as paid on October 1. For the 2-year period 2008 and 2009, the fund values and coupon payments were recorded in Table 4.1.

Table 4.1: Cash flows of fund

Time mm/dd/yy	Coupon received ($ millions)	Fund value before date ($ millions)
01/01/08		20.0
04/01/08	0.80	22.0
10/01/08	1.02	22.8
04/01/09	0.97	21.9
10/01/09	0.85	23.5
12/31/09		25.0

Calculate the TWRR and the DWRR of the fund.

Solution: As the coupon payments are withdrawals from the fund (the portfolio of bonds), the fund drops in value after the coupon payments. For example, the bond value drops to $22.0 - 0.80 = 21.2$ million on April 1, 2008 after the coupon payments. Thus, the TWRR is calculated as

$$R_T = \left[\frac{22}{20} \times \frac{22.80}{22.0 - 0.8} \times \frac{21.90}{22.80 - 1.02} \times \frac{23.50}{21.90 - 0.97} \times \frac{25.0}{23.50 - 0.85}\right]^{0.5} - 1 = 21.42\%.$$

To calculate the DWRR we solve R_D from the following equation

$$\begin{aligned} 20 &= \frac{0.8}{(1+R_D)^{0.25}} + \frac{1.02}{(1+R_D)^{0.75}} + \frac{0.97}{(1+R_D)^{1.25}} + \frac{0.85}{(1+R_D)^{1.75}} + \frac{25}{(1+R_D)^{2}} \\ &= 0.8v + 1.02v^3 + 0.97v^5 + 0.85v^7 + 25v^8, \end{aligned}$$

where $v = (1+R_D)^{-0.25}$. We let $(1+y)^{-1} = v$ and use Excel to obtain $y = 4.949\%$, so that the annual effective rate of return is $R_D = (1.04949)^4 - 1 = 21.31\%$. □

4.4 Portfolio Return

- We now consider the return of a portfolio of assets.

- Suppose a portfolio consists of N assets denoted by $A_1, \cdots, A_N$. Let the value of asset A_j in the portfolio at time 0 be A_{0j}, for $j = 1, \cdots, N$.

- We allow A_{0j} to be negative for some j, so that asset A_j is sold short in the portfolio.

- The portfolio value at time 0 is $B_0 = \sum_{j=1}^{N} A_{0j}$. Let the asset values at time 1 be A_{1j}, so that the portfolio value is $B_1 = \sum_{j=1}^{N} A_{1j}$.

- Denote R_P as the return of the portfolio in the period from time 0 to time 1. Thus,

$$R_P = \frac{B_1 - B_0}{B_0} = \frac{B_1}{B_0} - 1.$$

- We define

$$w_j = \frac{A_{0j}}{B_0},$$

which is the proportion of the value of asset A_j in the initial portfolio, so that

$$\sum_{j=1}^{N} w_j = 1,$$

and $w_j < 0$ if asset j is short sold in the portfolio.

- We also denote

$$R_j = \frac{A_{1j} - A_{0j}}{A_{0j}} = \frac{A_{1j}}{A_{0j}} - 1,$$

which is the rate of return of asset j. Thus,

$$\begin{aligned}
1 + R_P &= \frac{B_1}{B_0} \\
&= \frac{1}{B_0} \sum_{j=1}^{N} A_{1j} \\
&= \sum_{j=1}^{N} \frac{A_{0j}}{B_0} \times \frac{A_{1j}}{A_{0j}} \\
&= \sum_{j=1}^{N} w_j (1 + R_j),
\end{aligned}$$

which implies

$$R_P = \sum_{j=1}^{N} w_j R_j, \tag{4.13}$$

so that the return of the portfolio is the weighted average of the returns of the individual assets.

- Eq (4.13) is an identity, and applies to realized returns as well as returns as random variables. If we take the expectations of (4.13), we obtain

$$\mathrm{E}(R_P) = \sum_{j=1}^{N} w_j \mathrm{E}(R_j), \tag{4.14}$$

so that the expected return of the portfolio is equal to the weighted average of the expected returns of the component assets.

- The variance of the portfolio return is given by

$$\mathrm{Var}(R_P) = \sum_{j=1}^{N} w_j^2 \mathrm{Var}(R_j) + \underbrace{\sum_{h=1}^{N} \sum_{j=1}^{N}}_{h \neq j} w_h w_j \mathrm{Cov}(R_h,\, R_j). \tag{4.15}$$

- For example, consider a portfolio consisting of two funds, a stock fund and a bond fund, with returns denoted by R_S and R_B, respectively. Likewise, we use w_S and w_B to denote their weights in the portfolio.

- Then, we have

$$\mathrm{E}(R_P) = w_S \mathrm{E}(R_S) + w_B \mathrm{E}(R_B), \tag{4.16}$$

and

$$\mathrm{Var}(R_P) = w_S^2 \mathrm{Var}(R_S) + w_B^2 \mathrm{Var}(R_B) + 2 w_S w_B \mathrm{Cov}(R_S,\, R_B), \tag{4.17}$$

where $w_S + w_B = 1$.

Example 4.11: A stock fund has an expected return of 0.15 and variance of 0.0625. A bond fund has an expected return of 0.05 and variance of 0.0016. The correlation coefficient between the two funds is -0.2.

(a) What is the expected return and variance of the portfolio with 80% in the stock fund and 20% in the bond fund?

(b) What is the expected return and variance of the portfolio with 20% in the stock fund and 80% in the bond fund?

(c) How would you weight the two funds in your portfolio so that your portfolio has the lowest possible variance?

Solution: For (a), we use (4.16) and (4.17), with $w_S = 0.8$ and $w_B =$

0.2, to obtain

$$
\begin{aligned}
\mathrm{E}(R_P) &= (0.8)(0.15) + (0.2)(0.05) = 13\% \\
\mathrm{Var}(R_P) &= (0.8)^2(0.0625) + (0.2)^2(0.0016) + 2(0.8)(0.2)(-0.2)\sqrt{(0.0016)(0.0625)} \\
&= 0.03942.
\end{aligned}
$$

Thus, the portfolio has a standard deviation of $\sqrt{0.03942} = 19.86\%$. For (b), we do similar calculations, with $w_S = 0.2$ and $w_B = 0.8$, to obtain $\mathrm{E}(R_P) = 7\%$, $\mathrm{Var}(R_P) = 0.002884$ and a standard deviation of $\sqrt{0.002884} = 5.37\%$.

Hence, we observe that the portfolio with a higher weightage in stock has a higher expected return but also a higher standard deviation, i.e., higher risk.

For (c) we rewrite (4.17) as

$$\mathrm{Var}(R_P) = w_S^2 \mathrm{Var}(R_S) + (1 - w_S)^2 \mathrm{Var}(R_B) + 2w_S(1 - w_S)\mathrm{Cov}(R_S,\, R_B).$$

To minimize the variance, we differentiate $\mathrm{Var}(R_P)$ with respect to w_S to obtain

$$2w_S\mathrm{Var}(R_S) - 2(1 - w_S)\mathrm{Var}(R_B) + 2(1 - 2w_S)\mathrm{Cov}(R_S,\, R_B).$$

Equating the above to zero, we solve for w_S to obtain

$$\begin{aligned} w_S &= \frac{\mathrm{Var}(R_B) - \mathrm{Cov}(R_S,\, R_B)}{\mathrm{Var}(R_B) + \mathrm{Var}(R_S) - 2\mathrm{Cov}(R_S,\, R_B)} \\ &= 5.29\%. \end{aligned}$$

The expected return of this portfolio is 5.53%, its variance is 0.001410, and its standard deviation is 3.75%, which is lower than the standard deviation of the bond fund of 4%. Hence, this portfolio *dominates* the bond fund, in the sense that it has a higher expected return and a lower standard deviation.

Note that the fact that the above portfolio indeed minimizes the variance can be verified by examining the second-order condition. $\square$