

Financial Mathematics for Actuaries (Third Edition)

Chapter 2

Annuities

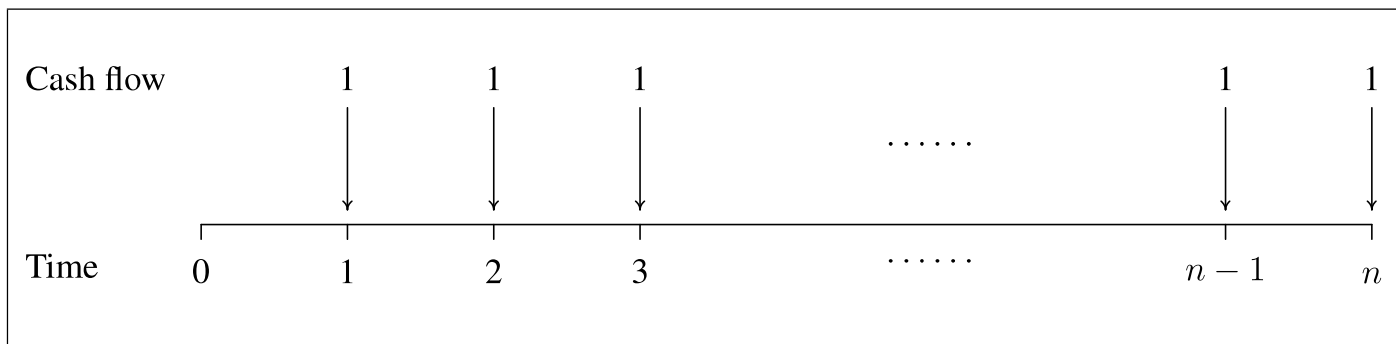
Learning Objectives

1. Annuity-immediate and annuity-due
2. Present and future values of annuities
3. Perpetuities and deferred annuities
4. Other accumulation methods
5. Payment periods and compounding periods
6. Varying annuities

2.1 Annuity-Immediate

- Consider an annuity with payments of 1 unit each, made at the end of every year for n years.
- This kind of annuity is called an **annuity-immediate** (also called an **ordinary annuity** or an **annuity in arrears**).
- The **present value of an annuity** is the sum of the present values of each payment.
- See Figure 2.1 for the time diagram of an annuity-immediate.

Figure 2.1: Time diagram of an annuity-immediate



Example 2.1: Calculate the present value of an annuity-immediate of amount \$100 paid annually for 5 years at the rate of interest of 9%.

Solution: Table 2.1 summarizes the present values of the payments as well as their total.

Table 2.1: Present value of annuity

Year	Payment (\$)	Present value (\$)
1	100	$100 (1.09)^{-1} = 91.74$
2	100	$100 (1.09)^{-2} = 84.17$
3	100	$100 (1.09)^{-3} = 77.22$
4	100	$100 (1.09)^{-4} = 70.84$
5	100	$100 (1.09)^{-5} = 64.99$
Total		388.97

□

- We are interested in the value of the annuity at time 0, called the present value, and the accumulated value of the annuity at time n , called the future value.

- Suppose the rate of interest per period is i , and we assume the compound-interest method applies.
- Let $a_{\overline{n}|i}$ denote the present value of the annuity, which is sometimes denoted as $a_{\overline{n}|}$ when the rate of interest is understood.
- As the present value of the j th payment is v^j , where $v = 1/(1+i)$ is the discount factor, the present value of the annuity is (see Appendix A.5 for the sum of a geometric progression)

$$\begin{aligned}
 a_{\overline{n}|} &= v + v^2 + v^3 + \cdots + v^n \\
 &= v \times \left[\frac{1 - v^n}{1 - v} \right] \\
 &= \frac{1 - v^n}{i} \\
 &= \frac{1 - (1+i)^{-n}}{i}.
 \end{aligned} \tag{2.1}$$

- The accumulated value of the annuity at time n is denoted by $s_{\overline{n}|i}$ or $s_{\overline{n}}$.
- This is the future value of $a_{\overline{n}}$ at time n . Thus, we have

$$\begin{aligned} s_{\overline{n}} &= a_{\overline{n}} \times (1 + i)^n \\ &= \frac{(1 + i)^n - 1}{i}. \end{aligned} \tag{2.2}$$

- If the annuity is of level payments of P , the present and future values of the annuity are $Pa_{\overline{n}}$ and $Ps_{\overline{n}}$, respectively.

Example 2.2: Calculate the present value of an annuity-immediate of amount \$100 paid annually for 5 years at the rate of interest of 9% using formula (2.1). Also calculate its future value at time 5.

Solution: From (2.1), the present value of the annuity is

$$100 a_{\overline{5}|} = 100 \times \left[\frac{1 - (1.09)^{-5}}{0.09} \right] = \$388.97,$$

which agrees with the solution of Example 2.1. The future value of the annuity is

$$(1.09)^5 \times (100 a_{\overline{5}|}) = (1.09)^5 \times 388.97 = \$598.47.$$

Alternatively, the future value can be calculated as

$$100 s_{\overline{5}|} = 100 \times \left[\frac{(1.09)^5 - 1}{0.09} \right] = \$598.47.$$

□

Example 2.3: Calculate the present value of an annuity-immediate of amount \$100 payable quarterly for 10 years at the annual rate of interest

of 8% convertible quarterly. Also calculate its future value at the end of 10 years.

Solution: Note that the rate of interest per payment period (quarter) is $(8/4)\% = 2\%$, and there are $4 \times 10 = 40$ payments. Thus, from (2.1) the present value of the annuity-immediate is

$$100 a_{\overline{40}|0.02} = 100 \times \left[\frac{1 - (1.02)^{-40}}{0.02} \right] = \$2,735.55,$$

and the future value of the annuity-immediate is

$$2735.55 \times (1.02)^{40} = \$6,040.20.$$

□

- A common problem in financial management is to determine the installments required to pay back a loan. We may use (2.1) to calculate the amount of level installments required.

Example 2.4: A man borrows a loan of \$20,000 to purchase a car at annual rate of interest of 6%. He will pay back the loan through monthly installments over 5 years, with the first installment to be made one month after the release of the loan. What is the monthly installment he needs to pay?

Solution: The rate of interest per payment period is $(6/12)\% = 0.5\%$. Let P be the monthly installment. As there are $5 \times 12 = 60$ payments, from (2.1) we have

$$\begin{aligned} 20,000 &= P a_{\overline{60}|0.005} \\ &= P \times \left[\frac{1 - (1.005)^{-60}}{0.005} \right] \\ &= P \times 51.7256, \end{aligned}$$

so that

$$P = \frac{20,000}{51.7256} = \$386.66.$$

□

- The example below illustrates the calculation of the required installment for a targeted future value.

Example 2.5: A man wants to save \$100,000 to pay for his son's education in 10 years' time. An education fund requires the investors to deposit equal installments annually at the end of each year. If interest of 7.5% is paid, how much does the man need to save each year in order to meet his target?

Solution: We first calculate $s_{\overline{10}|}$, which is equal to

$$\frac{(1.075)^{10} - 1}{0.075} = 14.1471.$$

Then the required amount of installment is

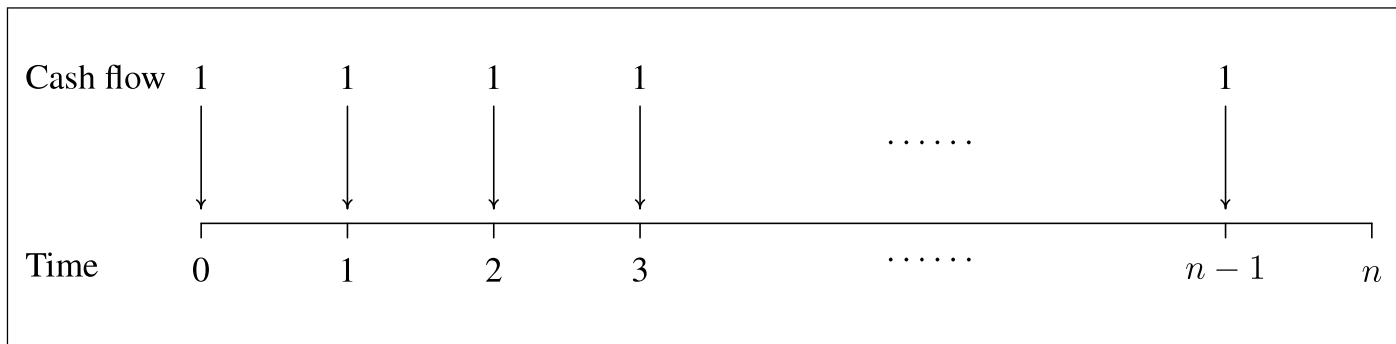
$$P = \frac{100,000}{s_{\overline{10}|}} = \frac{100,000}{14.1471} = \$7,068.59.$$

□

2.2 Annuity-Due

- An **annuity-due** is an annuity for which the payments are made at the beginning of the payment periods
- The first payment is made at time 0, and the last payment is made at time $n - 1$.
- See Figure 2.2 for the time diagram of an annuity-due.

Figure 2.2: Time diagram of an annuity-due



- We denote the present value of the annuity-due at time 0 by $\ddot{a}_{\overline{n}|i}$ (or $\ddot{a}_{\overline{n}|}$), and the future value of the annuity at time n by $\ddot{s}_{\overline{n}|i}$ (or $\ddot{s}_{\overline{n}|}$).
- The formula for $\ddot{a}_{\overline{n}|}$ can be derived as follows

$$\begin{aligned}
 \ddot{a}_{\overline{n}|} &= 1 + v + \dots + v^{n-1} \\
 &= \frac{1 - v^n}{1 - v} \\
 &= \frac{1 - v^n}{d}.
 \end{aligned} \tag{2.3}$$

- Also, we have

$$\begin{aligned}
 \ddot{s}_{\overline{n}|} &= \ddot{a}_{\overline{n}|} \times (1 + i)^n \\
 &= \frac{(1 + i)^n - 1}{d}.
 \end{aligned} \tag{2.4}$$

- As each payment in an annuity-due is paid one period ahead of the corresponding payment of an annuity-immediate, the present value of each payment in an annuity-due is $(1+i)$ times of the present value of the corresponding payment in an annuity-immediate. Hence,

$$\ddot{a}_{\overline{n}|} = (1+i) a_{\overline{n}|} \quad (2.5)$$

and, similarly,

$$\ddot{s}_{\overline{n}|} = (1+i) s_{\overline{n}|}. \quad (2.6)$$

- As an annuity-due of n payments consists of a payment at time 0 and an annuity-immediate of $n - 1$ payments, the first payment of which is to be made at time 1, we have

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \quad (2.7)$$

- Similarly, if we consider an annuity-immediate with $n + 1$ payments at time 1, 2, \dots , $n + 1$ as an annuity-due of n payments starting at time 1 plus a final payment at time $n + 1$, we can conclude

$$s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1. \quad (2.8)$$

Example 2.6: A company wants to provide a retirement plan for an employee who is aged 55 now. The plan will provide her with an annuity-immediate of \$7,000 every year for 15 years upon her retirement at the age of 65. The company is funding this plan with an annuity-due of 10 years. If the rate of interest is 5%, what is the amount of installment the company should pay?

Solution: We first calculate the present value of the retirement annuity.

This is equal to

$$7,000 a_{\overline{15}|} = 7,000 \times \left[\frac{1 - (1.05)^{-15}}{0.05} \right] = \$72,657.61.$$

This amount should be equal to the future value of the company's installments P , which is $P\ddot{s}_{\overline{10}|}$. Now from (2.4), we have

$$\ddot{s}_{\overline{10}|} = \frac{(1.05)^{10} - 1}{1 - (1.05)^{-1}} = 13.2068,$$

so that

$$P = \frac{72,657.61}{13.2068} = \$5,501.53.$$

2.3 Perpetuity, Deferred Annuity and Annuity Values at Other Times

- A **perpetuity** is an annuity with no termination date, i.e., $n \rightarrow \infty$.
- An example that resembles a perpetuity is the dividends of a preferred stock.
- To calculate the present value of a perpetuity, we note that, as $v < 1$, $v^n \rightarrow 0$ as $n \rightarrow \infty$. Thus, from (2.1), we have

$$a_{\overline{\infty}|} = \frac{1}{i}. \quad (2.9)$$

- For the case when the first payment is made immediately, we have, from (2.3),

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}. \quad (2.10)$$

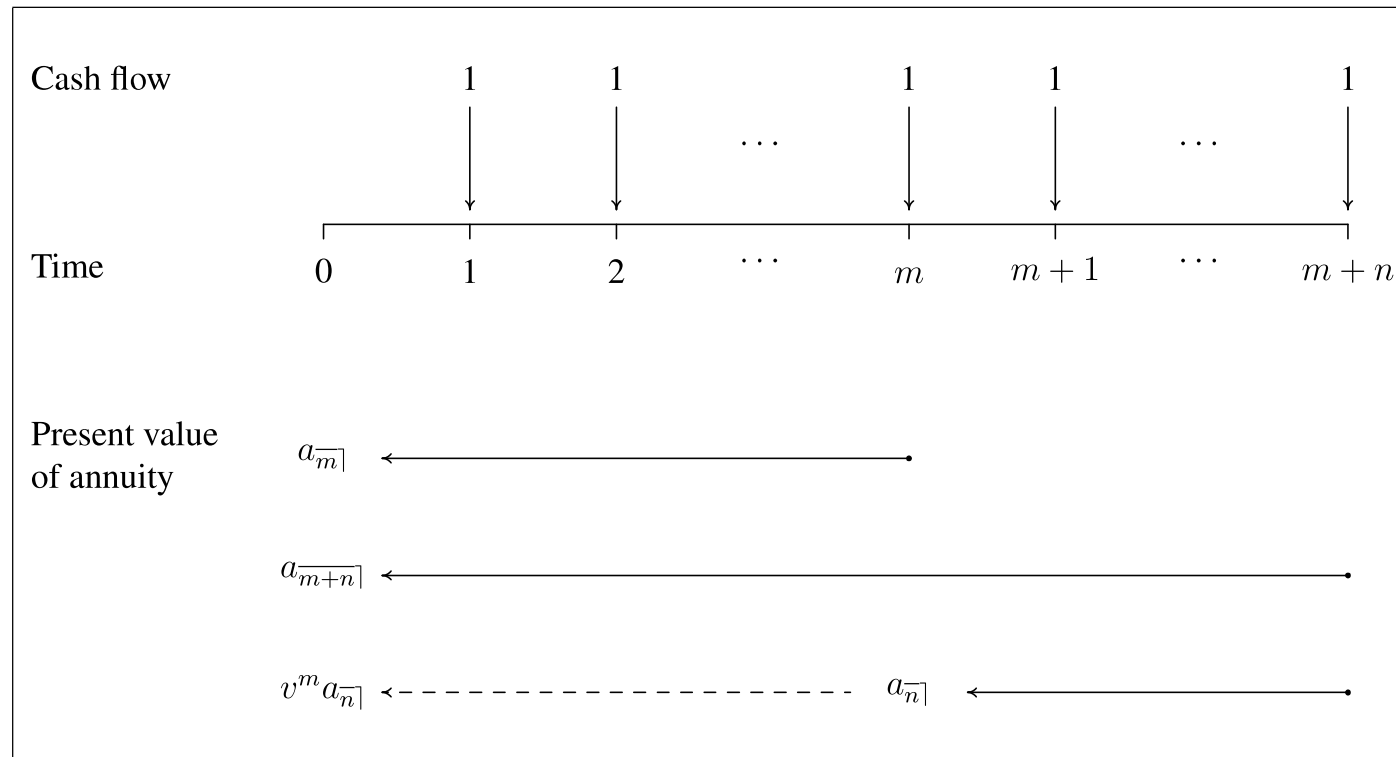
- A **deferred annuity** is one for which the first payment starts some time in the future.
- Consider an annuity with n unit payments for which the first payment is due at time $m + 1$.
- This can be regarded as an n -period annuity-immediate to start at time m , and its present value is denoted by ${}_m|a_{\overline{n}|i}$ (or ${}_m|a_{\overline{n}|}$ for short).

Thus, we have

$$\begin{aligned}
{}_m|a_{\overline{n}|} &= v^m a_{\overline{n}|} \\
&= v^m \times \left[\frac{1 - v^n}{i} \right] \\
&= \frac{v^m - v^{m+n}}{i} \\
&= \frac{(1 - v^{m+n}) - (1 - v^m)}{i} \\
&= a_{\overline{m+n}|} - a_{\overline{m}|}.
\end{aligned} \tag{2.11}$$

- To understand the above equation, see Figure 2.3.

Figure 2.3: Illustration of equation (2.11)



- From (2.11), we have

$$\begin{aligned} a_{\overline{m+n}|} &= a_{\overline{m}|} + v^m a_{\overline{n}|} \\ &= a_{\overline{n}|} + v^n a_{\overline{m}|}. \end{aligned} \tag{2.12}$$

- Multiplying the above equations throughout by $1 + i$, we have

$$\begin{aligned} \ddot{a}_{\overline{m+n}|} &= \ddot{a}_{\overline{m}|} + v^m \ddot{a}_{\overline{n}|} \\ &= \ddot{a}_{\overline{n}|} + v^n \ddot{a}_{\overline{m}|}. \end{aligned} \tag{2.13}$$

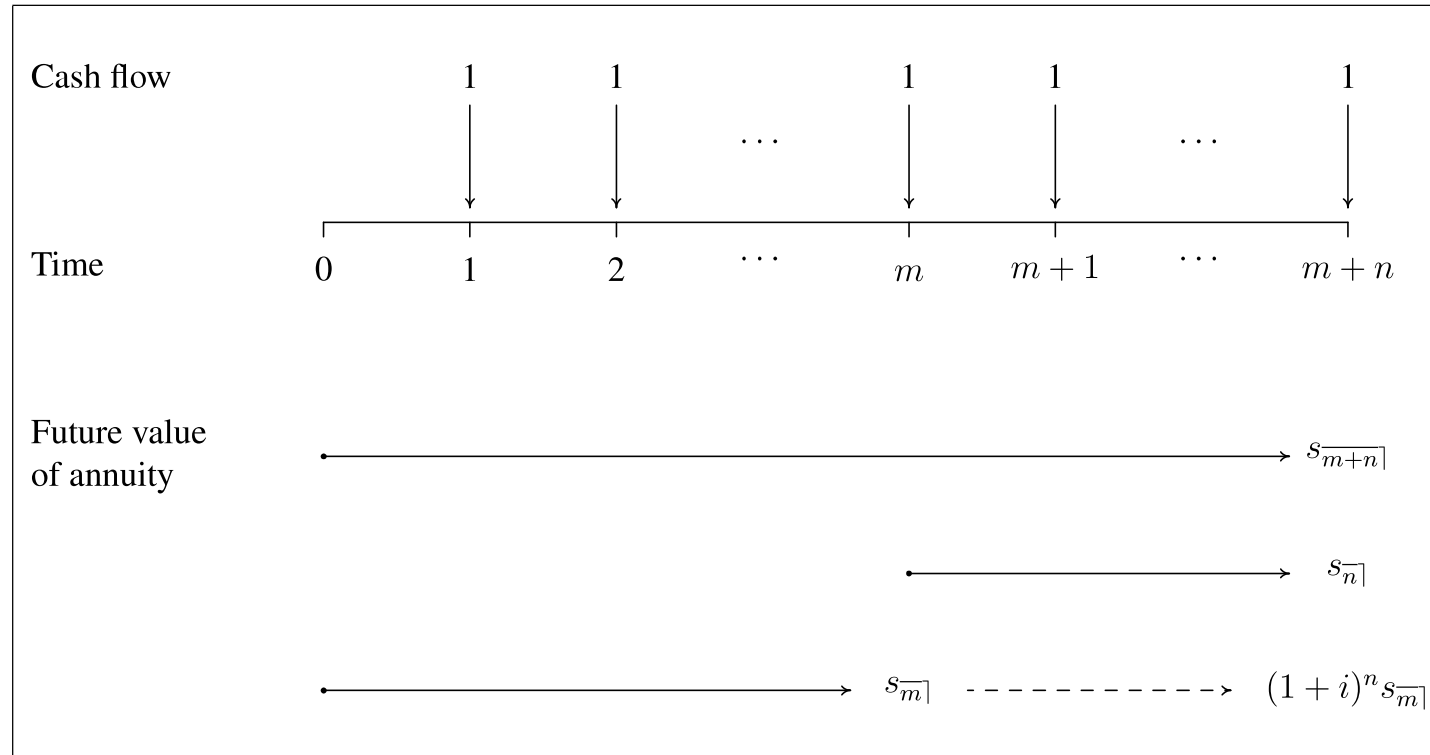
- We also denote $v^m \ddot{a}_{\overline{n}|}$ as ${}_m|\ddot{a}_{\overline{n}|}$, which is the present value of a n -payment annuity of unit amounts due at time $m, m+1, \dots, m+n-1$.
- If we multiply the equations in (2.12) throughout by $(1 + i)^{m+n}$, we

obtain

$$\begin{aligned} s_{\overline{m+n}|} &= (1+i)^n s_{\overline{m}|} + s_{\overline{n}|} \\ &= (1+i)^m s_{\overline{n}|} + s_{\overline{m}|}. \end{aligned} \tag{2.14}$$

- See Figure 2.4 for illustration.

Figure 2.4: Illustration of equation (2.14)



- It is also straightforward to see that

$$\begin{aligned}\ddot{s}_{\overline{m+n}|} &= (1+i)^n \ddot{s}_{\overline{m}|} + \ddot{s}_{\overline{n}|} \\ &= (1+i)^m \ddot{s}_{\overline{n}|} + \ddot{s}_{\overline{m}|}.\end{aligned}\tag{2.15}$$

- We now return to (2.2) and write it as

$$s_{\overline{m+n}|} = (1+i)^{m+n} a_{\overline{m+n}|},$$

so that

$$v^m s_{\overline{m+n}|} = (1+i)^n a_{\overline{m+n}|},\tag{2.16}$$

for arbitrary positive integers m and n .

- How do you interpret this equation?

2.4 Annuities Under Other Accumulation Methods

- We have so far discussed the calculations of the present and future values of annuities assuming compound interest.
- We now extend our discussion to other interest-accumulation methods.
- We consider a general accumulation function $a(\cdot)$ and assume that the function applies to any cash-flow transactions in the future.
- As stated in Section 1.7, any payment at time $t > 0$ starts to accumulate interest according to $a(\cdot)$ as a payment made at time 0.
- Given the accumulation function $a(\cdot)$, the present value of a unit payment due at time t is $1/a(t)$, so that the present value of a n -

period annuity-immediate of unit payments is

$$a_{\overline{n}|} = \sum_{t=1}^n \frac{1}{a(t)}. \quad (2.17)$$

- The future value at time n of a unit payment at time $t < n$ is $a(n - t)$, so that the future value of a n -period annuity-immediate of unit payments is

$$s_{\overline{n}|} = \sum_{t=1}^n a(n - t). \quad (2.18)$$

- If (1.35) is satisfied so that $a(n - t) = a(n)/a(t)$ for $n > t > 0$, then

$$s_{\overline{n}|} = \sum_{t=1}^n \frac{a(n)}{a(t)} = a(n) \sum_{t=1}^n \frac{1}{a(t)} = a(n) a_{\overline{n}|}. \quad (2.19)$$

- This result is satisfied for the compound-interest method, but not the simple-interest method or other accumulation schemes for which equation (1.35) does not hold.

Example 2.7: Suppose $\delta(t) = 0.02t$ for $0 \leq t \leq 5$, find $a_{\overline{5}|}$ and $s_{\overline{5}|}$.

Solution: We first calculate $a(t)$, which, from (1.26), is

$$\begin{aligned} a(t) &= \exp\left(\int_0^t 0.02s \, ds\right) \\ &= \exp(0.01t^2). \end{aligned}$$

Hence, from (2.17),

$$a_{\overline{5}|} = \frac{1}{e^{0.01}} + \frac{1}{e^{0.04}} + \frac{1}{e^{0.09}} + \frac{1}{e^{0.16}} + \frac{1}{e^{0.25}} = 4.4957,$$

and, from (2.18),

$$s_{\overline{5}|} = 1 + e^{0.01} + e^{0.04} + e^{0.09} + e^{0.16} = 5.3185.$$

Note that $a(5) = e^{0.25} = 1.2840$, so that

$$a(5) a_{\overline{5}|} = 1.2840 \times 4.4957 = 5.7724 \neq s_{\overline{5}|}.$$

□

- Note that in the above example, $a(n - t) = \exp[0.01(n - t)^2]$ and

$$\frac{a(n)}{a(t)} = \exp[0.01(n^2 - t^2)],$$

so that $a(n - t) \neq a(n)/a(t)$ and (2.19) does not hold.

Example 2.8: Calculate $a_{\overline{3}|}$ and $s_{\overline{3}|}$ if the nominal rate of interest is 5% per annum, assuming (a) compound interest, and (b) simple interest.

Solution: (a) Assuming compound interest, we have

$$a_{\overline{3}|} = \frac{1 - (1.05)^{-3}}{0.05} = 2.723,$$

and

$$s_{\overline{3}|} = (1.05)^3 \times 2.72 = 3.153.$$

(b) For simple interest, the present value is

$$a_{\overline{3}|} = \sum_{t=1}^3 \frac{1}{a(t)} = \sum_{t=1}^3 \frac{1}{1+rt} = \frac{1}{1.05} + \frac{1}{1.1} + \frac{1}{1.15} = 2.731,$$

and the future value at time 3 is

$$s_{\overline{3}|} = \sum_{t=1}^3 a(3-t) = \sum_{t=1}^3 (1+r(3-t)) = 1.10 + 1.05 + 1.0 = 3.150.$$

At the same nominal rate of interest, the compound-interest method generates higher interest than the simple-interest method. Therefore, the future value under the compound-interest method is higher, while its present value is lower. Also, note that for the simple-interest method, $a(3) a_{\overline{3}|} = 1.15 \times 2.731 = 3.141$, which is different from $s_{\overline{3}|} = 3.150$. \square

2.5 Payment Periods, Compounding Periods and Continuous Annuities

We now consider the case where the payment period differs from the interest-conversion period. We first consider the following two examples.

Example 2.9: Find the present value of an annuity-due of \$200 per quarter for 2 years, if interest is compounded monthly at the nominal rate of 8%.

Solution: This is the situation where the payments are made less frequently than interest is converted. We first calculate the effective rate of interest per quarter, which is

$$\left[1 + \frac{0.08}{12}\right]^3 - 1 = 2.01\%.$$

As there are $n = 8$ payments, the required present value is

$$200 \ddot{a}_{\overline{8}|0.0201} = 200 \times \left[\frac{1 - (1.0201)^{-8}}{1 - (1.0201)^{-1}} \right] = \$1,493.90.$$

□

Example 2.10: Find the present value of an annuity-immediate of \$100 per quarter for 4 years, if interest is compounded semiannually at the nominal rate of 6%.

Solution: This is the situation where payments are made more frequently than interest is converted. We first calculate the effective rate of interest per quarter, which is

$$\left[1 + \frac{0.06}{2} \right]^{\frac{1}{2}} - 1 = 1.49\%.$$

Thus, the required present value is

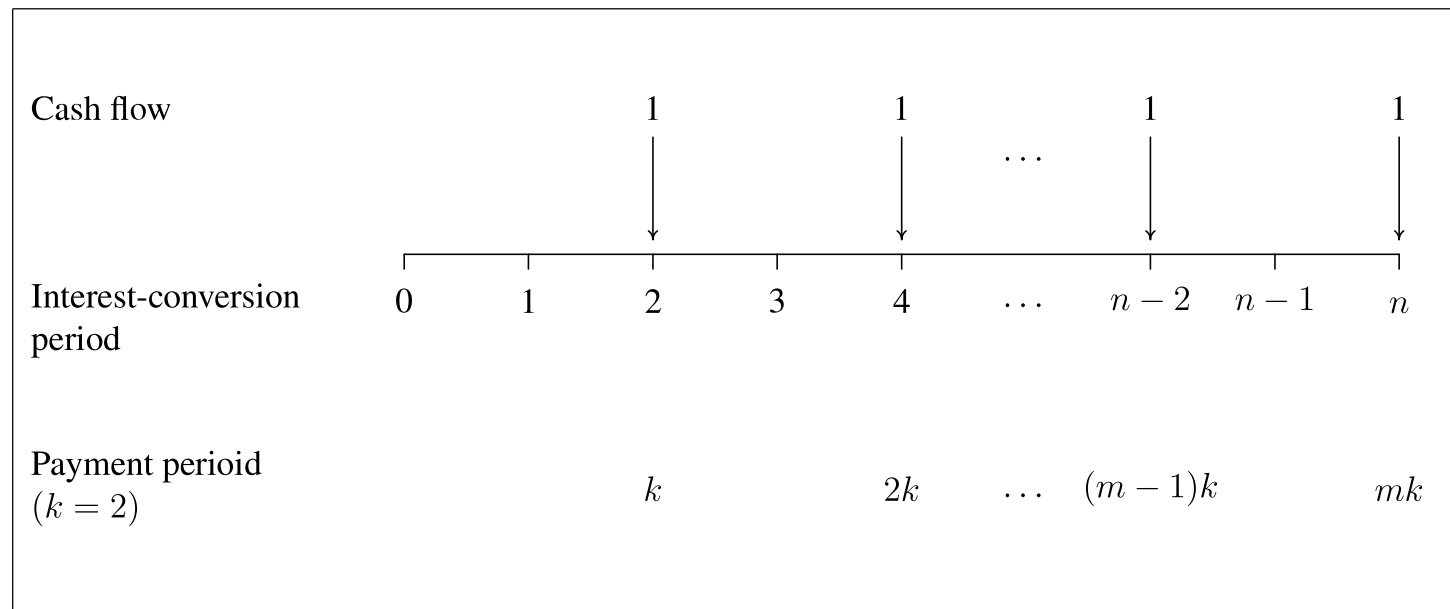
$$100 a_{\overline{16}|0.0149} = 100 \times \left[\frac{1 - (1.0149)^{-16}}{0.0149} \right] = \$1,414.27.$$

□

- It is possible to derive algebraic formulas to compute the present and future values of annuities for which the period of installment is different from the period of compounding.
- We first consider the case where payments are made less frequently than interest conversion, which occurs at time 1, 2, \dots , etc.
- Let i denote the effective rate of interest per interest-conversion period. Suppose a m -payment annuity-immediate consists of unit payments at time $k, 2k, \dots, mk$. We denote $n = mk$, which is the number of interest-conversion periods for the annuity.

- Figure 2.6 illustrates the cash flows for the case of $k = 2$.

Figure 2.6: Payments less frequent than interest conversion
($k = 2$)



- The present value of the above annuity-immediate is (we let $w = v^k$)

$$\begin{aligned}
v^k + v^{2k} + \dots + v^{mk} &= w + w^2 + \dots + w^m \\
&= w \times \left[\frac{1 - w^m}{1 - w} \right] \\
&= v^k \times \left[\frac{1 - v^n}{1 - v^k} \right] \\
&= \frac{1 - v^n}{(1 + i)^k - 1} \\
&= \frac{a_{\overline{n}|}}{s_{\overline{k}|}}, \tag{2.20}
\end{aligned}$$

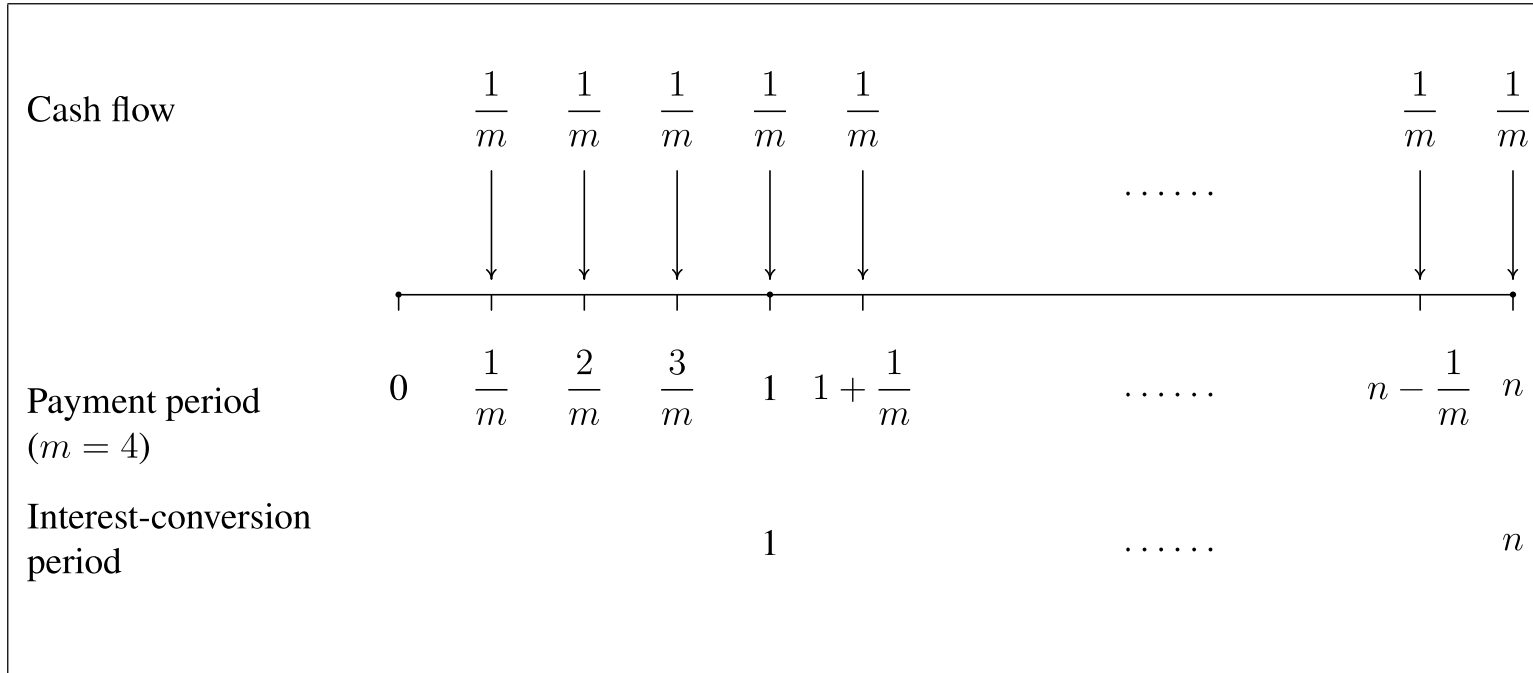
and the future value of the annuity is

$$(1 + i)^n \frac{a_{\overline{n}|}}{s_{\overline{k}|}} = \frac{s_{\overline{n}|}}{s_{\overline{k}|}}. \tag{2.21}$$

- We now consider the case where the payments are made more frequently than interest conversion.

- Let there be mn payments for an annuity-immediate occurring at time $1/m, 2/m, \dots, 1, 1 + 1/m, \dots, 2, \dots, n$, and let i be the effective rate of interest per interest-conversion period. Thus, there are mn payments over n interest-conversion periods.
- Suppose each payment is of the amount $1/m$, so that there is a nominal amount of unit payment in each interest-conversion period.
- Figure 2.7 illustrates the cash flows for the case of $m = 4$.

Figure 2.7: Payments more frequent than interest conversion
 ($m = 4$)



- We denote the present value of this annuity at time 0 by $a_{\overline{n}|i}^{(m)}$, which can be computed as follows (we let $w = v^{\frac{1}{m}}$)

$$\begin{aligned}
a_{\overline{n}|i}^{(m)} &= \frac{1}{m} \left(v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v + v^{1+\frac{1}{m}} + \cdots + v^n \right) \\
&= \frac{1}{m} (w + w^2 + \cdots + w^{mn}) \\
&= \frac{1}{m} \left[w \times \frac{1 - w^{mn}}{1 - w} \right] \\
&= \frac{1}{m} \left[v^{\frac{1}{m}} \times \frac{1 - v^n}{1 - v^{\frac{1}{m}}} \right] \\
&= \frac{1}{m} \left[\frac{1 - v^n}{(1 + i)^{\frac{1}{m}} - 1} \right] \\
&= \frac{1 - v^n}{r^{(m)}}, \tag{2.22}
\end{aligned}$$

where

$$r^{(m)} = m \left[(1 + i)^{\frac{1}{m}} - 1 \right] \quad (2.23)$$

is the equivalent nominal rate of interest compounded m times per interest-conversion period (see (1.19)).

- The future value of the annuity-immediate is

$$\begin{aligned} s_{\overline{n}|i}^{(m)} &= (1 + i)^n a_{\overline{n}|i}^{(m)} \\ &= \frac{(1 + i)^n - 1}{r^{(m)}} \\ &= \frac{i}{r^{(m)}} s_{\overline{n}|}. \end{aligned} \quad (2.24)$$

- The above equation parallels (2.22), which can also be written as

$$a_{\overline{n}|i}^{(m)} = \frac{i}{r^{(m)}} a_{\overline{n}|}.$$

- If the mn -payment annuity is due at time $0, 1/m, 2/m, \dots, n-1/m$, we denote its present value at time 0 by $\ddot{a}_{\overline{n}|}^{(m)}$, which is given by

$$\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} a_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} \times \left[\frac{1-v^n}{r^{(m)}} \right]. \quad (2.25)$$

- Thus, from (1.22) we conclude

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|}. \quad (2.26)$$

- The future value of this annuity at time n is

$$\ddot{s}_{\overline{n}|}^{(m)} = (1+i)^n \ddot{a}_{\overline{n}|}^{(m)} = \frac{d}{d^{(m)}} \ddot{s}_{\overline{n}|}. \quad (2.27)$$

- For deferred annuities, the following results apply

$${}_q|a_{\overline{n}|}^{(m)} = v^q a_{\overline{n}|}^{(m)}, \quad (2.28)$$

and

$${}_q|\ddot{a}_{\overline{n}|}^{(m)} = v^q \ddot{a}_{\overline{n}|}^{(m)}. \quad (2.29)$$

Example 2.11: Solve the problem in Example 2.9 using (2.20).

Solution: We first note that $i = 0.08/12 = 0.0067$. Now $k = 3$ and $n = 24$ so that from (2.20), the present value of the annuity-immediate is

$$\begin{aligned} 200 \times \frac{a_{\overline{24}|0.0067}}{s_{\overline{3}|0.0067}} &= 200 \times \left[\frac{1 - (1.0067)^{-24}}{(1.0067)^3 - 1} \right] \\ &= \$1,464.27. \end{aligned}$$

Finally, the present value of the annuity-due is

$$(1.0067)^3 \times 1,464.27 = \$1,493.90.$$

□

Example 2.12: Solve the problem in Example 2.10 using (2.22) and (2.23).

Solution: Note that $m = 2$ and $n = 8$. With $i = 0.03$, we have, from (2.23)

$$r^{(2)} = 2 \times [\sqrt{1.03} - 1] = 0.0298.$$

Therefore, from (2.22), we have

$$a_{\overline{8}|0.03}^{(2)} = \frac{1 - (1.03)^{-8}}{0.0298} = 7.0720.$$

As the total payment in each interest-conversion period is \$200, the required present value is

$$200 \times 7.0720 = \$1,414.27.$$

□

- Now we consider (2.22) again. Suppose the annuities are paid *continuously* at the rate of 1 unit per interest-conversion period over n periods. Thus, $m \rightarrow \infty$ and we denote the present value of this continuous annuity by $\bar{a}_{n|}$.
- As $\lim_{m \rightarrow \infty} r^{(m)} = \delta$, we have, from (2.22),

$$\bar{a}_{n|} = \frac{1 - v^n}{\delta} = \frac{1 - v^n}{\ln(1 + i)} = \frac{i}{\delta} a_{n|}. \quad (2.30)$$

- The present value of a n -period continuous annuity of unit payment per period with a deferred period of q is given by

$${}_q|\bar{a}_{n|} = v^q \bar{a}_{n|} = \bar{a}_{q+n|} - \bar{a}_{q|}. \quad (2.31)$$

- To compute the future value of a continuous annuity of unit payment

per period over n periods, we use the following formula

$$\bar{s}_{\overline{n}|} = (1 + i)^n \bar{a}_{\overline{n}|} = \frac{(1 + i)^n - 1}{\ln(1 + i)} = \frac{i}{\delta} s_{\overline{n}|}. \quad (2.32)$$

- We now generalize the above results to the case of a general accumulation function $a(\cdot)$. The present value of a continuous annuity of unit payment per period over n periods is

$$\bar{a}_{\overline{n}|} = \int_0^n v(t) dt = \int_0^n \exp\left(-\int_0^t \delta(s) ds\right) dt. \quad (2.33)$$

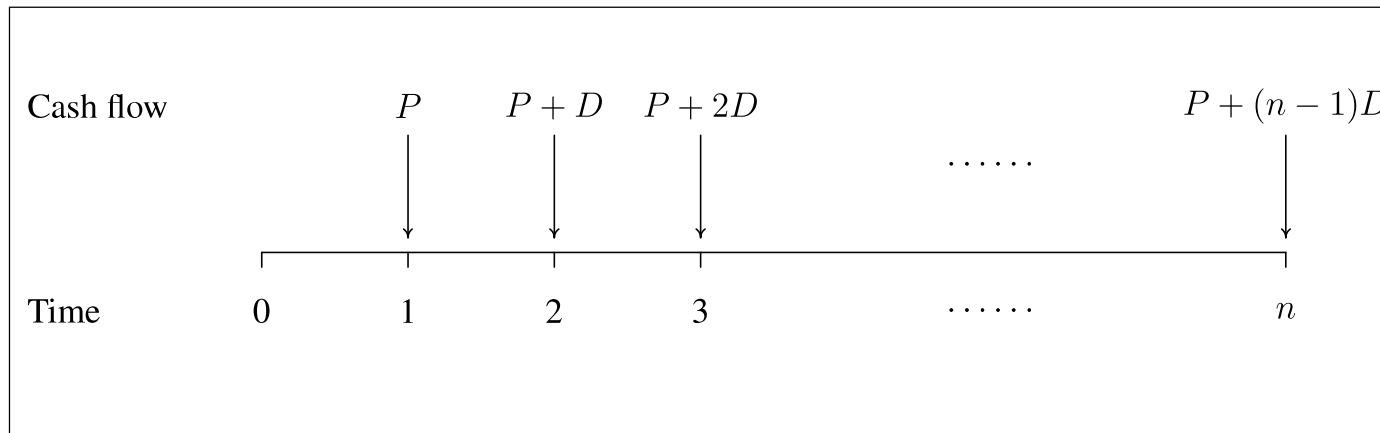
- To compute the future value of the annuity at time n , we assume that, as in Section 1.7, a unit payment at time t accumulates to $a(n - t)$ at time n , for $n > t \geq 0$.
- Thus, the future value of the annuity at time n is

$$\bar{s}_{\overline{n}|} = \int_0^n a(n - t) dt = \int_0^n \exp\left(\int_0^{n-t} \delta(s) ds\right) dt. \quad (2.34)$$

2.6 Varying Annuities

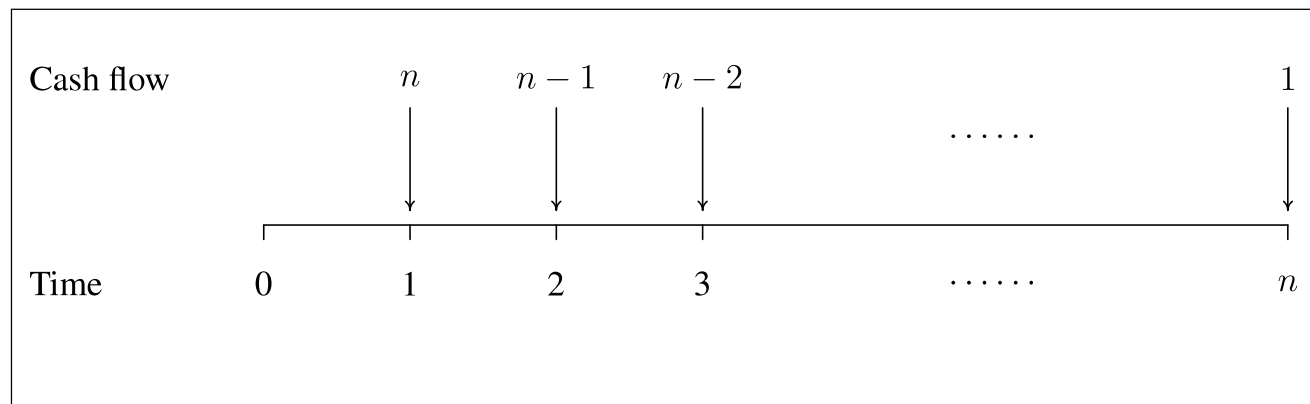
- We consider annuities the payments of which vary according to an arithmetic progression.
- Thus, we consider an annuity-immediate and assume the initial payment is P , with subsequent payments $P + D$, $P + 2D$, \dots , etc., so that the j th payment is $P + (j - 1)D$. See Figure 2.8.

Figure 2.8: Increasing annuity-immediate



- We allow D to be negative so that the annuity can be either stepping up or stepping down.
- However, for a n -payment annuity, $P + (n - 1)D$ must be positive so that negative cash flow is ruled out. See Figure 2.9.

Figure 2.9: Decreasing annuity-immediate, $P = n$ and $D = -1$



- We can see that the annuity can be regarded as the sum of the following annuities: (a) a n -period annuity-immediate with constant

amount P , and (b) $n - 1$ deferred annuities, where the j th deferred annuity is a $(n - j)$ -period annuity-immediate with level amount D to start at time j , for $j = 1, \dots, n - 1$.

- Thus, the present value of the varying annuity is

$$\begin{aligned}
 Pa_{\overline{n}|} + D \left[\sum_{j=1}^{n-1} v^j a_{\overline{n-j}|} \right] &= Pa_{\overline{n}|} + D \left[\sum_{j=1}^{n-1} v^j \frac{(1 - v^{n-j})}{i} \right] \\
 &= Pa_{\overline{n}|} + D \left[\frac{\left(\sum_{j=1}^{n-1} v^j \right) - (n-1)v^n}{i} \right] \\
 &= Pa_{\overline{n}|} + D \left[\frac{\left(\sum_{j=1}^n v^j \right) - nv^n}{i} \right] \\
 &= Pa_{\overline{n}|} + D \left[\frac{a_{\overline{n}|} - nv^n}{i} \right]. \tag{2.35}
 \end{aligned}$$

- For a n -period *increasing* annuity with $P = D = 1$, we denote its present and future values by $(Ia)_{\overline{n}|}$ and $(Is)_{\overline{n}|}$, respectively.

- It can be shown that

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \quad (2.36)$$

and

$$(Is)_{\overline{n}|} = \frac{s_{\overline{n+1}|} - (n+1)}{i} = \frac{\ddot{s}_{\overline{n}|} - n}{i}. \quad (2.37)$$

- For an increasing n -payment annuity-due with payments of $1, 2, \dots, n$ at time $0, 1, \dots, n-1$, the present value of the annuity is

$$(I\ddot{a})_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|}. \quad (2.38)$$

- This is the sum of a n -period level annuity-due of unit payments and a $(n-1)$ -payment increasing annuity-immediate with starting and incremental payments of 1.

- Thus, we have

$$(I\ddot{a})_{\overline{n}|} = \ddot{a}_{\overline{n}|} + (Ia)_{\overline{n-1}|}. \quad (2.39)$$

- For the case of a n -period *decreasing* annuity with $P = n$ and $D = -1$, we denote its present and future values by $(Da)_{\overline{n}|}$ and $(Ds)_{\overline{n}|}$, respectively.
- Figure 2.9 presents the time diagram of this annuity.
- It can be shown that

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \quad (2.40)$$

and

$$(Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}. \quad (2.41)$$

- We consider two types of increasing continuous annuities. First, we consider the case of a continuous n -period annuity with level payment (i.e., at a constant rate) of τ units from time $\tau - 1$ through time τ .

- We denote the present value of this annuity by $(I\bar{a})_{\overline{n}|}$, which is given by

$$(I\bar{a})_{\overline{n}|} = \sum_{\tau=1}^n \tau \int_{\tau-1}^{\tau} v^s ds = \sum_{\tau=1}^n \tau \int_{\tau-1}^{\tau} e^{-\delta s} ds. \quad (2.42)$$

- The above equation can be simplified to

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}. \quad (2.43)$$

- Second, we may consider a continuous n -period annuity for which the payment in the interval t to $t + \Delta t$ is $t\Delta t$, i.e., the instantaneous rate of payment at time t is t .

- We denote the present value of this annuity by $(\bar{I}\bar{a})_{\overline{n}|}$, which is given by

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n tv^t dt = \int_0^n te^{-\delta t} dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}. \quad (2.44)$$

- We now consider an annuity-immediate with payments following a geometric progression.
- Let the first payment be 1, with subsequent payments being $1 + k$ times the previous one. Thus, the present value of an annuity with

n payments is (for $k \neq i$)

$$\begin{aligned}
 v + v^2(1+k) + \cdots + v^n(1+k)^{n-1} &= v \sum_{t=0}^{n-1} [v(1+k)]^t \\
 &= v \sum_{t=0}^{n-1} \left[\frac{1+k}{1+i} \right]^t \\
 &= v \left[\frac{1 - \left(\frac{1+k}{1+i} \right)^n}{1 - \frac{1+k}{1+i}} \right] \\
 &= \frac{1 - \left(\frac{1+k}{1+i} \right)^n}{i - k}. \quad (2.45)
 \end{aligned}$$

Example 2.13: An annuity-immediate consists of a first payment of \$100, with subsequent payments increased by 10% over the previous one

until the 10th payment, after which subsequent payments decreases by 5% over the previous one. If the effective rate of interest is 10% per payment period, what is the present value of this annuity with 20 payments?

Solution: The present value of the first 10 payments is (use the second line of (2.45))

$$100 \times 10(1.1)^{-1} = \$909.09.$$

For the next 10 payments, $k = -0.05$ and their present value at time 10 is

$$100(1.10)^9(0.95) \times \frac{1 - \left(\frac{0.95}{1.1}\right)^{10}}{0.1 + 0.05} = 1,148.64.$$

Hence, the present value of the 20 payments is

$$909.09 + 1,148.64(1.10)^{-10} = \$1,351.94.$$

□

Example 2.14: An investor wishes to accumulate \$1,000 at the end of year 5. He makes level deposits at the beginning of each year for 5 years. The deposits earn a 6% annual effective rate of interest, which is credited at the end of each year. The interests on the deposits earn 5% effective interest rate annually. How much does he have to deposit each year?

Solution: Let the level annual deposit be A . The interest received at the end of year 1 is $0.06A$, which increases by $0.06A$ annually to $5 \times 0.06A$ at the end of year 5. Thus, the interest is a 5-payment increasing annuity with $P = D = 0.06A$, earning annual interest of 5%. Hence, we have the equation

$$1,000 = 5A + 0.06A(Is)_{\overline{5}|0.05}.$$

From (2.37) we obtain $(Is)_{\overline{5}|0.05} = 16.0383$, so that

$$A = \frac{1,000}{5 + 0.06 \times 16.0383} = \$167.7206.$$

2.7 Term of Annuity

- We now consider the case where the annuity period may not be an integer. We consider $a_{\overline{n+k}|}$, where n is an integer and $0 < k < 1$. We note that

$$\begin{aligned} a_{\overline{n+k}|} &= \frac{1 - v^{n+k}}{i} \\ &= \frac{(1 - v^n) + (v^n - v^{n+k})}{i} \\ &= a_{\overline{n}|} + v^{n+k} \left[\frac{(1 + i)^k - 1}{i} \right] \\ &= a_{\overline{n}|} + v^{n+k} s_{\overline{k}|}. \end{aligned} \tag{2.46}$$

- Thus, $a_{\overline{n+k}|}$ is the sum of the present value of a n -period annuity-immediate with unit amount and the present value of an amount $s_{\overline{k}|}$

paid at time $n + k$.

Example 2.15: A principal of \$5,000 generates income of \$500 at the end of every year at an effective rate of interest of 4.5% for as long as possible. Calculate the term of the annuity and discuss the possibilities of settling the last payment.

Solution: The equation of value

$$500 a_{\overline{n}|0.045} = 5,000$$

implies $a_{\overline{n}|0.045} = 10$. As $a_{\overline{13}|0.045} = 9.68$ and $a_{\overline{14}|0.045} = 10.22$, the principal can generate 13 regular payments. The investment may be paid off with an additional amount A at the end of year 13, in which case

$$500 s_{\overline{13}|0.045} + A = 5,000 (1.045)^{13},$$

which implies $A = \$281.02$, so that the last payment is $\$781.02$. Alternatively, the last payment B may be made at the end of year 14, which is

$$B = 281.02 \times 1.045 = \$293.67.$$

If we adopt the approach in (2.46), we solve k from the equation

$$\frac{1 - (1.045)^{-(13+k)}}{0.045} = 10,$$

which implies

$$(1.045)^{13+k} = \frac{1}{0.55},$$

from which we obtain

$$k = \frac{\ln\left(\frac{1}{0.55}\right)}{\ln(1.045)} - 13 = 0.58.$$

Hence, the last payment C to be paid at time 13.58 years is

$$C = 500 \times \left[\frac{(1.045)^{0.58} - 1}{0.045} \right] = \$288.32.$$

Note that $A < C < B$, which is as expected, as this follows the order of the occurrence of the payments. In principle, all three approaches are justified. \square

- Generally, the effective rate of interest cannot be solved analytically from the equation of value. Numerical methods must be used for this purpose.

Example 2.16: A principal of \$5,000 generates income of \$500 at the end of every year for 15 years. What is the effective rate of interest?

Solution: The equation of value is

$$a_{\overline{15}|i} = \frac{5,000}{500} = 10,$$

so that

$$a_{\overline{15}|i} = \frac{1 - (1 + i)^{-15}}{i} = 10.$$

A simple grid search provides the following results

i	$a_{\overline{15} i}$
0.054	10.10
0.055	10.04
0.056	9.97

A finer search provides the answer 5.556%. □

- The Excel Solver may be used to calculate the effective rate of interest in Example 2.16.

- The computation is illustrated in Exhibit 2.1.
- We enter a guessed value of 0.05 in Cell A1 in the Excel worksheet.
- The following expression is then entered in Cell A2:
 $(1 - (1 + A1)^{-15})/A1$, which computes $a_{\overline{15}|i}$ with i equal to the value at A1.
- We can also use the Excel function `RATE` to calculate the rate of interest that equates the present value of an annuity-immediate to a given value. Specifically, consider the equations

$$a_{\overline{n}|i} - A = 0 \quad \text{and} \quad \ddot{a}_{\overline{n}|i} - A = 0.$$

Given n and A , we wish to solve for i , which is the rate of interest per payment period of the annuity-immediate or annuity-due. The use of the Excel function `RATE` to compute i is described as follows:

Excel function: RATE(np,1,pv,type,guess)

np = n ,

pv = $-A$,

type = 0 (or omitted) for annuity-immediate, 1 for annuity-due

guess = starting value, set to 0.1 if omitted

Output = i , rate of interest per payment period of the annuity

- To use RATE to solve for Example 2.16 we key in the following:
“=RATE(15,1,-10)”.