Financial Mathematics for Actuaries (Third Edition)

Chapter 1

Interest Accumulation and Time Value of Money

Learning Objectives

1. Basic principles in calculation of interest accumulation
2. Simple and compound interest
3. Frequency of compounding
4. Effective rate of interest
5. Rate of discount
6. Present and future values of a single payment

1.1 Accumulation Function and Amount Function

- The sum of money borrowed is called the **principal**.
- The borrower compensates the lender by paying **interest**.
- At the end of the loan period the borrower pays the lender the **accumulated amount**, which is equal to the sum of the principal plus interest.
- We denote $A(t)$ as the accumulated amount at time t, called the **amount function**.
- $A(0)$ is the initial principal and

$$I(t) = A(t) - A(t-1) \tag{1.1}$$

is the interest incurred from time $t-1$ to time t, namely, in the tth period.

- For the special case of an initial principal of 1 unit, we denote the accumulated amount at time t by $a(t)$, which is called the **accumulation function**.

- If the initial principal is $A(0) = k$, then

$$A(t) = k \times a(t).$$

1.2 Simple and Compound Interest

- While theoretically there are numerous ways of calculating the interest, there are two methods which are commonly used in practice.

- These are the **simple-interest method** and the **compound-interest method**.

- For the simple-interest method the interest earned over a period of time is proportional to the length of the period.

- The interest incurred from time 0 to time t, for a principal of 1 unit, is $r \times t$, where r is the constant of proportion called the **rate of interest**.

- Hence the amount function for the simple-interest method is

$$a(t) = 1 + rt, \qquad \text{for } t \geq 0, \tag{1.2}$$

and

$$A(t) = A(0)a(t) = A(0)(1 + rt), \qquad \text{for } t \geq 0. \tag{1.3}$$

- The most commonly used base is the year, in which case the term **annual rate of interest** is used. We shall maintain this assumption, unless stated otherwise.

Example 1.1: A person borrows \$2,000 for 3 years at simple interest. The rate of interest is 8% per annum. What are the interest charges for years 1 and 2? What is the accumulated amount at the end of year 3?

Solution: The interest charges for years 1 and 2 are both equal to

$$2{,}000 \times 0.08 = \$160.$$

The accumulated amount at the end of year 3 is

$$2{,}000\,(1 + 0.08 \times 3) = \$2{,}480.$$

- For the compound-interest method the accumulated amount over a period of time is the principal for the next period.

- The amount function becomes

$$a(t) = (1 + r)^t, \qquad \text{for } t = 0, 1, 2, \cdots, \tag{1.4}$$

  and the accumulation function is

$$A(t) = A(0)a(t) = A(0)(1 + r)^t, \qquad \text{for } t = 0, 1, 2, \cdots. \tag{1.5}$$

- **Two remarks**

  – For the compound-interest method the accumulated amount at the end of a year becomes the principal for the following year. This is in contrast to the simple-interest method, for which the principal remains unchanged through time.

  – While (1.2) and (1.3) apply for $t \geq 0$, (1.4) and (1.5) hold only for integral $t \geq 0$. There are alternative ways to define the accumulation function for the compound-interest method when t is not an integer.

Example 1.2: Solve the problem in Example 1.1 using the compound-interest method.

Solution: The interest for year 1 is

$$2{,}000 \times 0.08 = \$160.$$

For year 2 the principal is

$$2{,}000 + 160 = \$2{,}160,$$

so that the interest for the year is

$$2{,}160 \times 0.08 = \$172.80.$$

The accumulated amount at the end of year 3 is

$$2{,}000\,(1 + 0.08)^3 = \$2{,}519.42.$$

$\square$

- Compounding has the effect of generating a larger accumulated amount.

- When the interest rate is high, compounding the interest induces the principal to grow much faster than the simple-interest method.

Table 1.1: Accumulated amount for a principal of $100

Year	5% interest		10% interest	
	Simple interest ($)	Compound interest ($)	Simple interest ($)	Compound interest ($)
1	105.00	105.00	110.00	110.00
2	110.00	110.25	120.00	121.00
3	115.00	115.76	130.00	133.10
4	120.00	121.55	140.00	146.41
5	125.00	127.63	150.00	161.05
6	130.00	134.01	160.00	177.16
7	135.00	140.71	170.00	194.87
8	140.00	147.75	180.00	214.36
9	145.00	155.13	190.00	235.79
10	150.00	162.89	200.00	259.37

1.3 Frequency of Compounding

- Although the rate of interest is often quoted in annual term, the interest accrued to an investment is often paid more frequently than once a year.

- For example, a savings account may pay interest at 3% per year, where the interest is credited monthly.

- The **frequency of interest payment** (also called the **frequency of compounding**) makes an important difference to the accumulated amount and the total interest earned. Thus, it is important to define the rate of interest accurately.

- We use $i^{(m)}$ to denote the **nominal rate of interest payable $\boldsymbol{m}$ times a year**. Thus, m is the frequency of compounding per year and $1/m$ year is the **compounding period** or **conversion period**.

- Assume t (in years) is an integer multiple of $1/m$, i.e., tm is an integer representing the number of interest-conversion periods over t years.

- The interest earned over the next $1/m$ year, i.e., the next interest-conversion period, is

$$a(t) \times i^{(m)} \times \frac{1}{m}, \qquad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \cdots.$$

Thus,

$$a(1) = a(0) + \frac{a(0)i^{(m)}}{m} = a(0)\left[1 + \frac{i^{(m)}}{m}\right] = 1 + \frac{i^{(m)}}{m},$$

and in general

$$a\left(t + \frac{1}{m}\right) = a(t)\left[1 + \frac{i^{(m)}}{m}\right], \qquad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \cdots,$$

so that

$$a(t) = \left[1 + \frac{i^{(m)}}{m}\right]^{mt}, \qquad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \cdots, \tag{1.6}$$

and

$$A(t) = A(0)\left[1 + \frac{i^{(m)}}{m}\right]^{mt}, \qquad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \cdots. \tag{1.7}$$

Example 1.3: A person deposits \$1,000 into a savings account that earns 3% interest payable monthly. How much interest will be credited in the first month? What is the accumulated amount at the end of the first month?

Solution: The rate of interest over one month is

$$0.03 \times \frac{1}{12} = 0.25\%,$$

so that the interest earned over one month is

$$1{,}000 \times 0.0025 = \$2.50,$$

and the accumulated amount after one month is

$$1{,}000 + 2.50 = \$1{,}002.50.$$

□

Example 1.4: \$1,000 is deposited into a savings account that pays 3% interest with monthly compounding. What is the accumulated amount after two and a half years? What is the amount of interest earned over this period?

Solution: The investment interval is 30 months. Thus, using (1.7), the accumulated amount is

$$1{,}000 \left[1 + \frac{0.03}{12}\right]^{30} = \$1{,}077.78.$$

The amount of interest earned over this period is

$$1{,}077.78 - 1{,}000 = \$77.78.$$

$\square$

- When the loan period is not an integer multiple of the compounding period, we may extend (1.6) and (1.7) to apply to any $tm \geq 0$ (not necessarily an integer).

- At the same nominal rate of interest, the more frequent the interest is paid, the faster the accumulated amount grows. For example, assuming the nominal rate of interest to be 5% and the principal to be \$1,000, the accumulated amounts after 1 year under several different compounding frequencies are given in Table 1.2.

Table 1.2: Accumulated amount for a principal of \$1,000

Frequency of interest payment	m	Accumulated amount (\$)
Yearly	1	1,050.00
Quarterly	4	1,050.95
Monthly	12	1,051.16
Daily	365	1,051.27

- When the compounding frequency m increases, the accumulated amount tends to a limit.

- Let $\bar{i}$ denote the nominal rate of interest for which compounding is made over infinitely small intervals (i.e., $m \to \infty$ so that $\bar{i} = i^{(\infty)}$).

- We call this compounding scheme **continuous compounding**.

- For practical purposes, daily compounding is very close to continuous compounding.

From the well-known limit theorem (see Appendix A.1) that

$$\lim_{m\to\infty}\left[1+\frac{\bar{i}}{m}\right]^m = e^{\bar{i}} \tag{1.8}$$

for any constant $\bar{i}$, we conclude that, for continuous compounding, the accumulation function (see (1.6)) is

$$a(t) = \lim_{m\to\infty}\left[1+\frac{\bar{i}}{m}\right]^{mt} = \left[\lim_{m\to\infty}\left(1+\frac{\bar{i}}{m}\right)^m\right]^t = e^{\bar{i}t}. \tag{1.9}$$

- We call $\bar{i}$ the **continuously compounded rate of interest**.

- Equation (1.9) provides the accumulation function of the continuously compounding scheme at nominal rate of interest $\bar{i}$.

1.4 Effective Rate of Interest

- The accumulated amount depends on the compounding frequency. Comparing two investment schemes by just referring to their nominal rates of interest without taking into account their compounding frequencies may be misleading.

- The measure called the **effective rate of interest** is often used to compare investments with different periods of compounding.

- The annual effective rate of interest for year t, which we denote by $i(t)$, is the ratio of the amount of interest earned in a year, from time $t-1$ to time t, to the accumulated amount at the beginning of the year (i.e., at time $t-1$).

- It can be calculated by the following formula

$$i(t) = \frac{I(t)}{A(t-1)} = \frac{A(t) - A(t-1)}{A(t-1)} = \frac{a(t) - a(t-1)}{a(t-1)}. \qquad (1.10)$$

- For the simple-interest method, we have

$$i(t) = \frac{(1 + rt) - (1 + r(t-1))}{1 + r(t-1)} = \frac{r}{1 + r(t-1)},$$

which decreases when t increases.

- For the compound-interest method with annual compounding (i.e., $m = 1$), we have

$$i(t) = \frac{(1 + i^{(1)})^t - (1 + i^{(1)})^{t-1}}{(1 + i^{(1)})^{t-1}} = i^{(1)},$$

which is the nominal rate of interest and does not vary with t.

- When m-compounding is used, the effective rate of interest is

$$i(t) = \frac{\left[1+\dfrac{i^{(m)}}{m}\right]^{tm} - \left[1+\dfrac{i^{(m)}}{m}\right]^{(t-1)m}}{\left[1+\dfrac{i^{(m)}}{m}\right]^{(t-1)m}} = \left[1+\frac{i^{(m)}}{m}\right]^{m} - 1, \quad (1.11)$$

which again does not vary with t.

- When $m > 1$,

$$\left[1+\frac{i^{(m)}}{m}\right]^{m} - 1 > i^{(m)},$$

so that the effective rate of interest is larger than the nominal rate of interest.

- For continuous compounding, we have

$$i(t) = \frac{\exp(\bar{i}t) - \exp[\bar{i}(t-1)]}{\exp[\bar{i}(t-1)]} = e^{\bar{i}} - 1, \qquad (1.12)$$

which again does not vary with t.

- As the effective rate of interest for the compound-interest method does not vary with t, we shall simplify the notation and denote $i \equiv i(t)$.

Example 1.7: Consider two investment schemes A and B. Scheme A offers 12% interest with annual compounding. Scheme B offers 11.5% interest with monthly compounding. Calculate the effective rates of interest of the two investments. Which scheme would you choose?

Solution: The effective rate of interest of Scheme A is equal to its nominal rate of interest, i.e., 12%. The effective rate of interest of Scheme B is

$$\left[1 + \frac{0.115}{12}\right]^{12} - 1 = 12.13\%.$$

Although Scheme A has a higher nominal rate of interest, Scheme B offers a higher effective rate of interest. Hence, while an investment of \$100 in Scheme A will generate an interest of \$12 after one year, a similar investment in Scheme B will generate an interest of \$12.13 over the same period. Thus, Scheme B is preferred. $\square$

Another advantage of the effective rate of interest is that, for investments that extend beyond one year the calculation of the accumulated amount can be based on the effective rate without reference to the nominal rate.

Example 1.8: For the investment schemes in Example 1.7, calculate the accumulated amount after 10 years on a principal of \$1,000.

Solution: The accumulated amount after 10 years for Scheme A is

$$1{,}000\,(1 + 0.12)^{10} = \$3{,}105.85,$$

and that for Scheme B is

$$1{,}000\,(1 + 0.1213)^{10} = \$3{,}142.09.$$

Note that in the above example, the accumulated amount of Scheme B is calculated without making use of its nominal rate.

- While $i(t)$ defined in (1.10) is a 1-period effective rate, the concept can be generalized to a n-period effective rate.

- We denote $i(t, t + n)$ as the annual effective rate of interest in the period t to $t + n$, where integral $n > 1$.

- The amount $a(t)$ at time t compounded annually at the rate of $i(t, t+ n)$ per year accumulates to $a(t + n)$ at time $t + n$, where

$$a(t + n) = a(t)[1 + i(t, t + n)]^n, \tag{1.13}$$

which implies

$$i(t, t+n) = \left[\frac{a(t+n)}{a(t)} \right]^{\frac{1}{n}} - 1. \tag{1.14}$$

- (1.10) can also be applied to intervals of less than one year. Suppose $\Delta t < 1$, the effective rate of interest in the period t to $t + \Delta t$, denoted by $i(t, t + \Delta t)$ is defined as

$$i(t, t+\Delta t) = \frac{a(t+\Delta t) - a(t)}{a(t)}, \tag{1.15}$$

which is an effective rate over a period of Δt. Note that this rate is *not annualized*.

1.5 Rates of Discount

- Some financial transactions have interest paid or deducted up-front.

- A popular way of raising a short-term loan is to sell a financial security at a price less than the face value. Upon the maturity of the loan, the face value is repaid. For example, a **sale and repurchase** agreement (**REPO**) is a discount instrument.

- For a discount security, the shortfall between the sale price and the face value is called the **discount**.

- The **nominal principal** (**face value**) has to be adjusted to take account of the interest deducted, which affects the rate of interest, as opposed to the **rate of discount** that is quoted.

- If the loan period is one year, the **effective principal** of the loan after the interest is deducted is

Effective principal = Nominal principal × (1 − Rate of discount).

- The effective principal is $A(0)$ and the nominal principal is $A(1)$.
- If the quoted rate of discount is d, we have

$$A(0) = A(1)(1 - d), \tag{1.16}$$

and

$$I(1) = A(1) - A(0) = A(1)d.$$

- The equivalent effective rate of interest i over the period of the discount instrument is

$$i = \frac{A(1) - A(0)}{A(0)} = \frac{A(1)d}{A(1)(1 - d)} = \frac{d}{1 - d}, \tag{1.17}$$

from which we have

$$d = \frac{i}{1+i}.$$

- Combining (1.4) and (1.17), we can see that

$$\begin{aligned} a(t) &= (1+i)^t \\ &= \left[1 + \frac{d}{1-d}\right]^t \\ &= (1-d)^{-t}, \end{aligned}$$

which is the accumulated value of 1 at time t at the rate of discount d.

- When the loan period is less than 1 year, we should first calculate the rate of interest over the period of the loan and then calculate the effective rate of interest using the principle of compounding.

- Suppose the period of loan is $1/m$ year, we denote the **nominal rate of discount** by $d^{(m)}$.

- The nominal principal is $A(1/m)$ and the interest deducted is

$$I\left(\frac{1}{m}\right) = A\left(\frac{1}{m}\right) \times d^{(m)} \times \frac{1}{m},$$

so that the effective principal is

$$A(0) = A\left(\frac{1}{m}\right) - I\left(\frac{1}{m}\right) = A\left(\frac{1}{m}\right)\left[1 - \frac{d^{(m)}}{m}\right],$$

and the rate of interest charged over the $(1/m)$-year period as

$$\frac{A\left(\frac{1}{m}\right) - A(0)}{A(0)} = \frac{\frac{d^{(m)}}{m}}{1 - \frac{d^{(m)}}{m}} = \frac{d^{(m)}}{m - d^{(m)}}.$$

- Hence, the **annualized equivalent nominal rate of interest** is

$$i^{(m)} = m \times \frac{d^{(m)}}{m - d^{(m)}} = \frac{d^{(m)}}{1 - \dfrac{d^{(m)}}{m}}, \tag{1.18}$$

and the annual effective rate of interest is

$$i = \left[1 + \frac{i^{(m)}}{m}\right]^m - 1 = \left[1 + \frac{d^{(m)}}{m - d^{(m)}}\right]^m - 1 = \left[1 - \frac{d^{(m)}}{m}\right]^{-m} - 1. \tag{1.19}$$

- To compute the accumulation function $a(t)$ for a discount instrument with maturity of $1/m$ year ($m > 1$), we note

$$a(t) = (1 + i)^t = \left[1 - \frac{d^{(m)}}{m}\right]^{-mt}, \qquad \text{for } t = 0, \frac{1}{m}, \frac{2}{m}, \cdots. \tag{1.20}$$

- From (1.19) we have

$$1 - \frac{d^{(m)}}{m} = (1+i)^{-\frac{1}{m}}, \tag{1.21}$$

so that from (1.18) we conclude

$$i^{(m)} = (1+i)^{\frac{1}{m}} d^{(m)}. \tag{1.22}$$

Example 1.11: The discount rate of a 3-month REPO is 6% per annum. What is the annual effective rate of interest? What is the accumulated value of 1 in 2 years?

Solution: The rate of interest charged for the 3-month period is

$$\frac{0.06 \times \frac{1}{4}}{1 - 0.06 \times \frac{1}{4}} = 1.52\%.$$

Therefore, the equivalent nominal rate of interest compounded quarterly is

$$4 \times 0.0152 = 6.08\%,$$

and the annual effective rate of interest is

$$(1.0152)^4 - 1 = 6.22\%.$$

The accumulated value of 1 in 2 years is, using (1.20),

$$a(2) = \left[1 - \frac{0.06}{4}\right]^{-8} = 1.13.$$

Note that this can also be calculated as

$$(1 + i)^2 = (1.0622)^2 = 1.13.$$

□

1.6 Force of Interest

- If we divide $i(t, t + \Delta t)$ by Δt, we obtain the rate of interest of the investment *per unit time* in the interval $(t,\ t + \Delta t)$.

- The *instantaneous* rate, which is obtained when Δt is infinitesimally small, is

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{i(t, t + \Delta t)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{a(t + \Delta t) - a(t)}{a(t)} \right] \\
&= \frac{1}{a(t)} \lim_{\Delta t \to 0} \left[\frac{a(t + \Delta t) - a(t)}{\Delta t} \right] \\
&= \frac{a'(t)}{a(t)}, \qquad (1.24)
\end{aligned}
$$

where $a'(t)$ is the derivative of $a(t)$ with respect to t.

- Thus, we define
$$\delta(t) = \frac{a'(t)}{a(t)}, \tag{1.25}$$
which is called the **force of interest**.

- The force of interest is the instantaneous rate of increase of the accumulated amount, $a'(t)$, as a percentage of the accumulated amount at time t, $a(t)$.

- Given $a(t)$, the force of interest $\delta(t)$ can be computed using (1.25).

- Now we note that (1.25) can be written as
$$\delta(t) = \frac{d \ln a(t)}{dt},$$

from which we have

$$
\begin{aligned}
\int_0^t \delta(s)\,ds &= \int_0^t d\ln a(s) \\
&= \ln a(s)\big]_0^t \\
&= \ln a(t) - \ln a(0) \\
&= \ln a(t),
\end{aligned}
$$

so that

$$a(t) = \exp\left(\int_0^t \delta(s)\,ds\right). \tag{1.26}$$

- In the case when the force of interest is constant (not varying with t), we denote $\delta(t) \equiv \delta$, and we have

$$a(t) = e^{\delta t}. \tag{1.27}$$

- Thus, if the force of interest is constant, it is equal to the continuously compounded rate of interest, i.e., $\bar{i} = \delta$.

- For the simple-interest method, we obtain

$$a'(t) = r,$$

so that

$$\delta(t) = \frac{r}{1+rt}, \qquad \text{for } t \geq 0. \tag{1.28}$$

- Hence, $\delta(t)$ decreases as t increases.

- For the compound-interest method, we have

$$a'(t) = (1+i)^t \ln(1+i),$$

so that

$$\delta(t) = \frac{(1+i)^t \ln(1+i)}{(1+i)^t} = \ln(1+i). \tag{1.29}$$

- Thus, $\delta(t)$ does not vary with time and we write $\delta(t) \equiv \delta$ so that

$$e^{\delta} = 1 + i.$$

Example 1.13: A fund accumulates at a simple-interest rate of 5%. Another fund accumulates at a compound-interest rate of 4%, compounded yearly. When will the force of interest be the same for the two funds? After this point, which fund will have a higher force of interest?

Solution: From (1.28), the force of interest of the simple-interest fund at time t is

$$\delta(t) = \frac{0.05}{1 + 0.05t}.$$

From (1.29), the force of interest of the compound-interest fund is $\ln(1.04)$ at any time. The two funds have the same force of interest when

$$\frac{0.05}{1 + 0.05t} = \ln(1.04),$$

i.e.,

$$t = \frac{0.05 - \ln(1.04)}{0.05\ln(1.04)} = 5.4967,$$

after which the force of interest of the simple-interest fund remains lower than that of the compound-interest fund. □

Example 1.14: If a fund accumulates at force of interest $\delta(t) = 0.02t$, find the annual effective rate of interest over 2 years and 5 years.

Solution: From (1.26), we have

$$a(2) = \exp\left(\int_0^2 0.02s\,ds\right) = \exp\left(0.01s^2\big]_0^2\right) = e^{0.04}.$$

We solve the equivalent annual effective rate of interest i from the equation (compare this with (1.13))

$$(1+i)^2 = e^{0.04}$$

to obtain

$$i = e^{0.02} - 1 = 2.02\%.$$

Similarly,

$$a(5) = \exp\left(0.01s^2\big]_0^5\right) = e^{0.25},$$

so that

$$(1+i)^5 = e^{0.25}$$

and

$$i = e^{0.05} - 1 = 5.13\%.$$

$\square$

1.7 Present and Future Values

- At the effective rate of interest i, a 1-unit investment today will accumulate to $(1 + i)$ units at the end of the year.

- The accumulated amount $(1 + i)$ is also called the **future value** of 1 at the end of the year.

- Similarly, the future value of 1 at the end of year t is $(1 + i)^t$.

- A $(1/(1+i))$-unit payment invested today will accumulate to 1 unit at the end of the year.

- Thus, $1/(1 + i)$ is called the **present value** of 1 to be paid at the end of year 1.

- The present value of 1 to be paid at the end of year t is $1/(1 + i)^t$.

Example 1.15: Given $i = 6\%$, calculate the present value of 1 to be paid at (a) the end of year 1, (b) the end of year 5 and (c) 6.5 years.

Solution: (a) The present value of 1 to be paid at the end of year 1 is

$$\frac{1}{1+0.06} = 0.9434.$$

The answers to (b) and (c) are, respectively,

$$\frac{1}{(1+0.06)^5} = 0.7473$$

and

$$\frac{1}{(1+0.06)^{6.5}} = 0.6847.$$

$\square$

Example 1.16: An insurance agent offers a policy that pays a lump sum of \$50,000 five years later. If the rate of interest is 8%, how much would you pay for the plan?

Solution: The *fair* amount to pay for this policy is the present value of the lump sum, which is equal to

$$\frac{50{,}000}{(1.08)^5} = \$34{,}029.16.$$

$\square$

Example 1.17: A person wants to accumulate \$100,000 eight years from today to sponsor his son's education. If an investment plan offers him 8% compounded monthly, what amount must he invest today?

Solution: We first calculate the effective rate of interest, which is equal to

$$\left[1+\frac{0.08}{12}\right]^{12}-1=8.30\%.$$

The amount required today is thus

$$\frac{100{,}000}{(1.083)^8}=\$52{,}841.16.$$

$\square$

- We now denote

$$v=\frac{1}{1+i}, \tag{1.30}$$

  which is the **present value** of 1 to be paid 1 year later.

- It is also called the **discount factor.**

- Combining (1.17) and (1.30), we obtain

$$d=iv, \tag{1.31}$$

so that the present value of i is d.

- Also,

$$v + d = \frac{1}{1+i} + \frac{i}{1+i} = 1, \tag{1.32}$$

which says that a unit payment at time 1 is the sum of its present value and discount.

- Since $a(1) = 1 + i$, (1.30) implies

$$v = \frac{1}{a(1)},$$

and, for a general time t, we denote $v(t)$ as the present value of 1 to be paid at time t.

- Then

$$v(t) = \frac{1}{(1+i)^t} = \frac{1}{a(t)}, \tag{1.33}$$

which is the discount factor for payments at time t.

- For a general accumulation function $a(\cdot)$, the discount factor for payments at time t is

$$v(t) = \frac{1}{a(t)}.$$

Example 1.18: Find the sum of the present values of two payments of \$100 each to be paid at the end of year 4 and 9, if (a) interest is compounded semiannually at the nominal rate of 8% per year, and (b) the simple-interest method at 8% per year is used.

Solution: We first calculate the discount factors $v(4)$ and $v(9)$. For case (a), the effective rate of interest is

$$(1.04)^2 - 1 = 0.0816,$$

so that

$$v(4) = \frac{1}{(1.0816)^4} = 0.7307$$

and

$$v(9) = \frac{1}{(1.0816)^9} = 0.4936.$$

Hence, the present value of the two payments is

$$100(0.7307 + 0.4936) = \$122.43.$$

For case (b), we have

$$v(4) = \frac{1}{1 + 0.08 \times 4} = 0.7576$$

and

$$v(9) = \frac{1}{1 + 0.08 \times 9} = 0.5814,$$

so that the present value of the two payments is

$$100(0.7576 + 0.5814) = \$133.90.$$

$\square$

- We now consider a payment of 1 at a future time τ. What is the future value of this payment at time $t > \tau$?

- The answer to this question depends on how a new payment at a future time accumulates with interest.

- Let us assume that any future payment starts to accumulate interest following the same accumulation function as a payment made at time 0.

- As the 1-unit payment at time τ earns interest over a period of $t - \tau$ until time t, its accumulated value at time t is $a(t - \tau)$.

- However, if we consider a different scenario in which the 1-unit amount at time τ has been accumulated from time 0 and is not a new investment, what is the future value of this amount at time t?

- To answer this question, we first determine the invested amount at time 0, which is the present value of 1 due at time τ, i.e., $1/a(\tau)$. The future value of this investment at time t is then given by

$$\frac{1}{a(\tau)} \times a(t) = \frac{a(t)}{a(\tau)}.$$

- We can see that the future values of the two investments, i.e., a 1-unit investment at time τ versus a 1-unit amount at time τ accumulated from time 0, are not necessarily the same.

- However, they are equal if

$$a(t-\tau) = \frac{a(t)}{a(\tau)} \tag{1.35}$$

for $t > \tau > 0$.

- The compound-interest accumulation satisfies the condition (1.35).

- The principal at time τ accumulates interest at the rate of i per year, whether the principal is invested at time τ or is accumulated from the past. Specifically, we have

$$\frac{a(t)}{a(\tau)} = \frac{(1+i)^t}{(1+i)^\tau} = (1+i)^{t-\tau} = a(t-\tau).$$

- In contrast, simple-interest accumulation does not satisfy condition (1.35). The future value at time t of a unit payment at time τ is

$a(t-\tau) = 1 + r(t-\tau)$. However, we have

$$\frac{a(t)}{a(\tau)} = \frac{1+rt}{1+r\tau} = \frac{1+r\tau+r(t-\tau)}{1+r\tau} = 1+\frac{r(t-\tau)}{1+r\tau} < 1+r(t-\tau) = a(t-\tau).$$

Example 1.19: Let $a(t) = 0.02t^2 + 1$. Calculate the future value of an investment at $t = 5$ consisting of a payment of 1 now and a payment of 2 at $t = 3$.

Solution: The future value at time 5 is

$$a(5) + 2 \times a(2) = [0.02(5)^2 + 1] + 2[0.02(2)^2 + 1] = 3.66.$$

$\square$

Example 1.20: Let $\delta(t) = 0.01t$. Calculate the future value of an investment at $t = 5$ consisting of a payment of 1 now and a payment of 2 at $t = 3$.

Solution: We first derive the accumulation function from the force of interest, which is

$$a(t) = \exp\left(\int_0^t 0.01s\,ds\right) = \exp(0.005t^2).$$

Thus, $a(5) = \exp(0.125) = 1.1331$ and $a(3) = \exp(0.045) = 1.0460$, from which we obtain the future value of the investment as

$$a(5) + 2 \times a(2) = 1.1331 + 2(1.0460) = 3.2251.$$

$\square$

1.8 Equation of Value

- Consider a stream of cash flows occurring at different times. The present value of the cash flows is equal to the sum of the present values of each payment.

- Assume the payments are of values C_j occurring at time $j = 0, 1, \cdots, n$.

- If the annual effective rate of interest is i with corresponding discount factor v, the present value P of the cash flows is given by

$$P = \sum_{j=0}^{n} C_j v^j. \tag{1.36}$$

- The future value F of the cash flows at time n is

$$F = (1+i)^n P = (1+i)^n \left(\sum_{j=0}^{n} C_j v^j \right) = \sum_{j=0}^{n} C_j (1+i)^{n-j}. \tag{1.37}$$

- We call these the **equations of value**.

Example 1.21: At the annual effective rate of interest i, when will an initial principal be doubled?

Solution: This requires us to solve for n from the equation (note that $C_j = 0$ for $j > 0$, and we let $C_0 = 1$ and $F = 2$)

$$(1 + i)^n = 2,$$

from which

$$n = \frac{\log(2)}{\log(1 + i)}.$$

Thus, n is generally not an integer but can be solved exactly from the above equation.

To obtain an approximate solution for n, we note that $\ln(2) = 0.6931$ so that

$$n = \frac{0.6931}{i} \times \frac{i}{\ln(1+i)}.$$

We approximate the last fraction in the above equation by taking $i = 0.08$ to obtain

$$n \simeq \frac{0.6931}{i} \times 1.0395 = \frac{0.72}{i}.$$

Thus, n can be calculated approximately by dividing 0.72 by the effective rate of interest. This is called the **rule of 72**. It provides a surprisingly accurate approximation to n for a wide range of values of i. For example, when $i = 2\%$, the approximation gives $n = 36$ while the exact value is 35. When $i = 14\%$, the approximate value is 5.14 while the exact value is 5.29. $\square$

Example 1.22: How long will it take for \$100 to accumulate to \$300 if interest is compounded quarterly at the nominal rate of 6% per year?

Solution: Over one quarter, the interest rate is 1.5%. With $C_0 = \$100$, $C_j = 0$ for $j > 0$, and $F = \$300$, from (1.36) the equation of value is (n is in quarters)

$$100(1.015)^n = 300,$$

so that

$$n = \frac{\ln(3)}{\ln(1.015)} = 73.79 \text{ quarters},$$

i.e., 18.45 years. □

- The examples above concern only one payment. When multiple payments are involved, numerical methods may be required to calculate n. However, analytical formulas are available when the payments are level, and this will be discussed in the next chapter.

- A related problem is the solution of the rate of interest that will give rise to a targeted present value or future value with corresponding cash flows. The example below illustrates this point.

Example 1.24: A savings fund requires the investor to pay an equal amount of installment each year for 3 years, with the first installment to be paid immediately. At the end of the 3 years, a lump sum will be paid back to the investor. If the effective interest rate is 5%, what is the amount of the installment so that the investor can get back \$10,000?

Solution: Let k be the installment. From (1.37), the equation of value is

$$10{,}000 = k[(1.05)^3 + (1.05)^2 + 1.05] = 3.31k,$$

so that the installment is

$$k = \frac{10{,}000}{3.31} = \$3{,}021.03.$$

Table 1.3: Summary of accumulated value and present value formulas (all rates quoted per year)

Accumulation method	Freq of conversion per year	Rate of interest or discount	Future value of 1 at time t (in years): $a(t)$	Present value of 1 due at time t (in years): $1/a(t)$	Equations in book	Remarks
Compound interest	1	$i^{(1)} = i$	$(1+i)^t$	$(1+i)^{-t}$	(1.4)	also applies to non-integer $t > 0$
Compound interest	m	$i^{(m)}$	$\left[1 + \frac{i^{(m)}}{m}\right]^{mt}$	$\left[1 + \frac{i^{(m)}}{m}\right]^{-mt}$	(1.6)	$m > 1$ if compounding is more frequent than annually
Compound interest	∞	$\bar{i}$	$e^{\bar{i}t}$	$e^{-\bar{i}t}$	(1.9)	$\bar{i}$ is the constant force of interest δ
Compound discount	1	d	$(1-d)^{-t}$	$(1-d)^t$	(1.4), (1.7)	
Compound discount	m	$d^{(m)}$	$\left[1 - \frac{d^{(m)}}{m}\right]^{-mt}$	$\left[1 - \frac{d^{(m)}}{m}\right]^{mt}$	(1.19)	$m > 1$ for loan period shorter than 1 year
Simple interest		r	$1 + rt$	$(1+rt)^{-1}$	(1.2)	
Simple discount		d	$(1-dt)^{-1}$	$1 - dt$	(1.23)	